Aggregation of stochastic models

Problem presented by

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Dstl

Problem statement

Dstl simulate battles by means of stochastic evolution codes. These involve detailed simulation of many units in the battlefield. The Study Group was asked to investigate methods of 'aggregation' that would simulate a battle in simpler terms by aggregating the units together, so that simulations could be accelerated.

The Study Group proposed partitioning the battlefield into zones and treating the numbers of combatants in those zones as continuous variables obeying ordinary differential equations, possibly with stochastic terms. The parameters in those equations, and the stochastic terms, would need to be determined by running small, unit-to-unit, combats in the full simulator, which can be done much more quickly than simulating the whole battle.

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1 The Dstl aggregation problem

Dstl examines battle simulations based on stochastic evolution codes. One such code, known as `SIMBAT` (SIMple BATtle program) models the evolution of the battle as a sequence of turns in which the two sets of combating units move in a landscape. The units have objectives and act accordingly, they can acquire enemy units if lines-of-sight in the landscape permit this, and they can fire upon and disable enemy units with a certain probability. Based on the setting of a large number of parameters, and also on the outcome of pseudo-random decisions and engagements made in the course of the action, a final outcome of the battle is achieved. Figure 1 shows a `SIMBAT` screenshot.

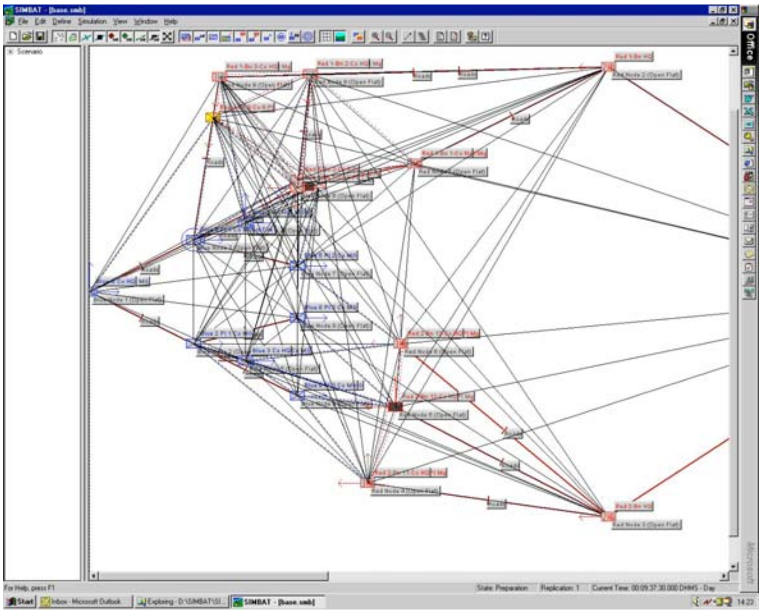

Figure 1: Screenshot from a `SIMBAT` battle. The setting is that of a 'standard' battle (see below).

The problem posed to the Study Group is the following: Is it possible to analyse and subsequently calculate a battle in terms of smaller subunits which can then be aggregated into the whole in a systematic fashion? This could potentially speed up the processing of a large number of simulations.

2 The concept of effective zones

Asked for a specific test example, Dstl pointed to what will be referred here to as a 'standard battle'. Two sets of units are to move and engage on an otherwise featureless rectangle as in Figure 2. The objective for the Blue units is to reach the point marked X. The objective for the Red units is to prevent this from taking place.

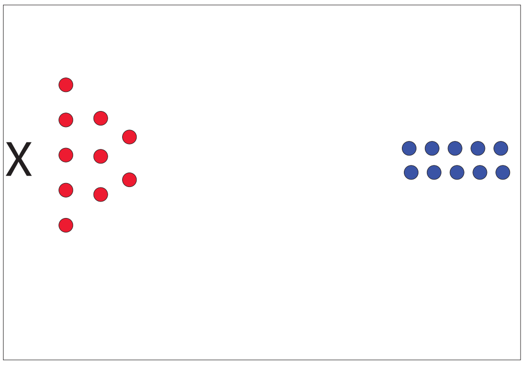

Figure 2: A 'standard' battle: Blue vs Red. The objective for Blue is to reach the point marked X. The objective for Red is to prevent that.

The tactic for Blue is to first initiate a diversion along one side of the rectangle (say the bottom side), followed then by a main charge along the opposite side, against the now (presumed) displaced defensive force of Red, as in Figure 3.

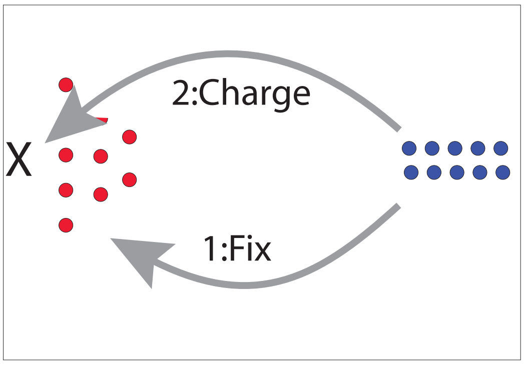

Figure 3: Tactics for the 'standard' battle.

In the spirit of reducing complicated statistical mechanics problems using mean-field theories, the Study Group introduced a concept of aggregation units as *effective zones*, as in Figure 4. Each zone is a part of the battle (not necessarily topologically connected, or in other words, not geographically continuous) in which one can identify a behaviour simple enough to be modelled by a set of (ordinary, deterministic) differential equations. For the standard battle, it suffices to consider 4 zones: Zone 1 which describes the 'charge' effort, Zone 2 which describes the 'fix' effort, Zone 3 which describes the defence against the 'fix' effort and Zone 4 which describes the defence against the 'charge' effort.

A small number of continuous variables describe the essential properties of each aggregation unit, and coupled differential equations describe the dynamics taking place between the zones. For example, the variable M_{B1} describes the 'mass' of Blue forces in

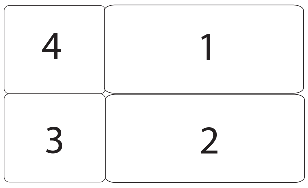

Figure 4: Effective Zones: the entire combat region is partitioned into a small number of zones each of which contains well defined activity by all groups. The 'standard' battle contains four such zones.

Zone 1, and the evolution of this variable is coupled to the variable M_{R4} describing the mass of Red forces in Zone 4. The simplest type of equation gives the rate change (loss) of M_{B1} as proportional to the product of M_{B1} and M_{R4}:

$$\dot{M}_{B1} = -k_{41} P_{R4B1} M_{R4} M_{B1} \tag{1}$$

where k_{41} is a positive constant describing 'combat efficiency' and P_{R4B1} is an auxiliary dynamical variable describing the 'perceived strength', *i.e.* the strength of the Red forces in Zone 4 as perceived by the Blue forces in Zone 1. Equation (1) is different from the standard Lanchester equations, but this is because Blue wants to get mass out of Zone 1 and into Zone 4. More general equations can be devised to incorporate alternative tactics or to illuminate particular issues of interest.

The combined system of ODEs describing M_{B1}, M_{B2}, M_{R3} and M_{R4}, as well as P_{B1R4}, P_{R4B1}, P_{B2R3} and P_{R3B2}, is now:

$$\begin{aligned}
\dot{M}_{B1} &= -K_{41}(P_{R4B1}, M_{R4}, M_{B1}) && (2)\\
\dot{M}_{B2} &= -K_{23}(P_{R3B2}, M_{R3}, M_{B2}) && (3)\\
\dot{M}_{R3} &= -K_{32}(P_{B2R3}, M_{R3}, M_{B2}) \\
&\quad -T_{34}(M_{R3}, M_{R4})M_{R3} + T_{43}(M_{R3}, M_{R4})M_{R4} && (4)\\
\dot{M}_{R4} &= -K_{41}(P_{B1R4}, M_{R4}, M_{B1}) \\
&\quad +T_{34}(M_{R3}, M_{R4})M_{R3} - T_{43}(M_{R3}, M_{R4})M_{R4} && (5)\\
\dot{P}_{B1R4} &= c_{11}(1 - P_{B1R4}) - r_{11}\Theta(P_{B1R4}M_{R4} - M_{B1}) && (6)\\
\dot{P}_{B2R3} &= c_{22}(1 - P_{B2R3}) && (7)\\
P_{R4B1} &= P_{B1R4} && (8)\\
P_{R3B2} &= P_{B2R3}. && (9)
\end{aligned}$$

Here $T_{34}(M_{R3}, M_{R4}) = \Theta(M_{R3} - 2M_{R4})e_{34}$ and $T_{43}(M_{R3}, M_{R4}) = \Theta(2M_{R4} - M_{R3})e_{43}$ involve the Heaviside function Θ and reflect the instantaneous troop 'masses' in zones 3 and 4. The function K_{ab} is taken to be simply $K_{ab}(P_{ab}, M_a, M_b) = k_{ab}P_{ab}M_aM_b$.

The parameter c_{ab} measures the rate of approach of troops in line-of-sight (related to troop movement), the parameter e_{ij} measures the rate of troop exchange between zones i and j, and the parameter r_{ij} denotes the rate of withdrawal of the troops in line-of-sight from zone i to zone j.

3 Dynamics of aggregation variables

The model (2-9) was implemented as a MATLAB routine and the solution curves examined. Figure 5 shows the output of two distinct simulations. In one, the parameters are set

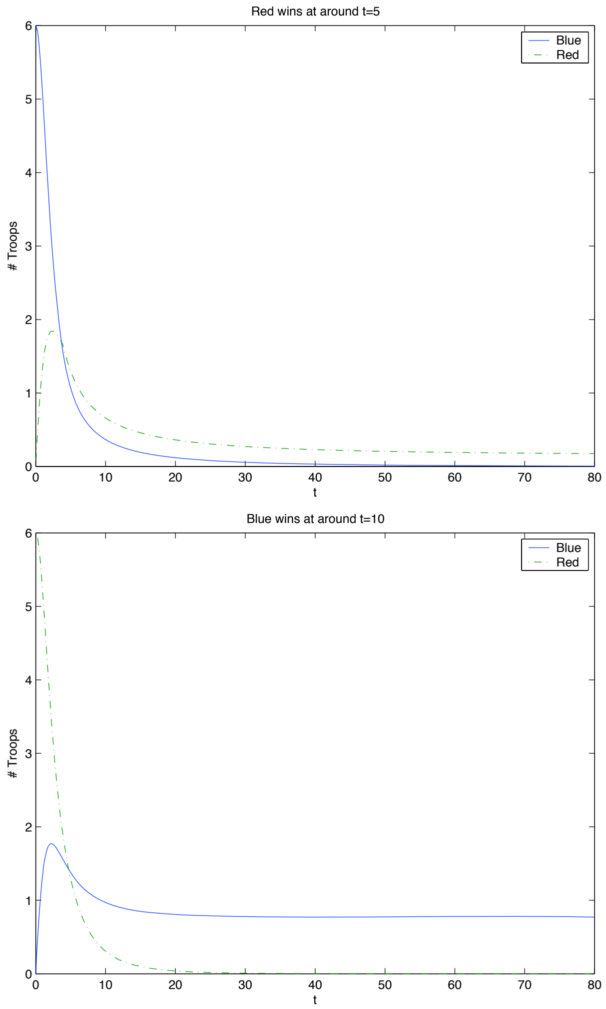

Figure 5: Evolution of troop 'masses' according to system (2–9) and parameters such that Red wins (above) or Blue wins (below).

so that Red wins, in the the other the parameters are set so that Blue wins. It is even possible to set parameters so that no one wins (*i.e.* both mass functions go to zero) as in Figure 6.

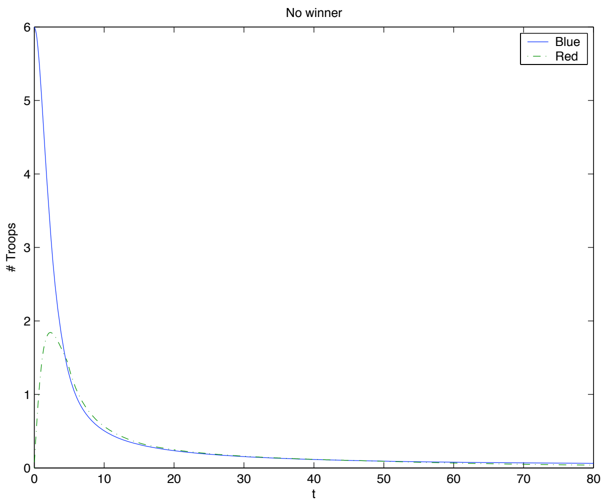

Figure 6: Parameters set to values so that no one wins: both masses go to zero.

4 Adding logistic units

To see how well this concept permits an increased level of detail, we added to the simple battle the further structure that a *part* of the blue force in, say, Zone 1 acts as a 'logistic' unit, in the sense of supply troops which will increase the combat effectiveness of friendly troops (through supplies) but do not actively partake in battle.

This logistic unit is described by a new dependent variable M_{B1L}, and the ODE system is augmented accordingly:

$$
\begin{aligned}
\dot{M}_{B1} &= -K_{41}(P_{R4B1}, M_{R4}, M_{B1}) && (10)\\
\dot{M}_{B1L} &= -K_{41L}(P_{R4B1L}, M_{R4}, M_{B1}, M_{B1L}) && (11)\\
\dot{M}_{B2} &= -K_{23}(P_{R3B2}, M_{R3}, M_{B2}) && (12)\\
\dot{M}_{R3} &= -K_{32}(P_{B2R3}, M_{R3}, M_{B2}) \\
&\quad -T_{34}(M_{R3}, M_{R4})M_{R3} + T_{43}(M_{R3}, M_{R4})M_{R4} && (13)\\
\dot{M}_{R4} &= -K_{41}(P_{B1R4}, M_{R4}, M_{B1}, M_{B1L}) \\
&\quad +T_{34}(M_{R3}, M_{R4})M_{R3} - T_{43}(M_{R3}, M_{R4})M_{R4} && (14)\\
\dot{P}_{B1R4} &= c_{11}(1 - P_{B1R4}) - r_{11}\Theta(P_{B1R4}M_{R4} - M_{B1}) && (15)\\
\dot{P}_{B1LR4} &= c_{11L}(c_L - P_{B1R4}) - r_{11L}\Theta(P_{B1R4}M_{R4} - M_{B1}) && (16)\\
\dot{P}_{B2R3} &= c_{22}(1 - P_{B2R3}) && (17)\\
P_{R4B1} &= P_{B1R4} && (18)\\
P_{R4B1L} &= P_{B1R4L} && (19)\\
P_{R3B2} &= P_{B2R3}. && (20)
\end{aligned}
$$

The mass of the logistic Blue unit in Zone 1 has a rate change which is a function of the perceived strength of opposing forces (we could assume that a large perceived strength will act to increase the mass of logistic troops), and also a function of relevant troop masses.

Also, a variable P_{B1LR4} is introduced — the perceived strength of the logistic Blue units in Zone 1 as perceived by the opposing Red forces in Zone 4. The change of this variable is related to the perceived strength of regular B1 troops. Notice that a sufficiently large value of the M_{R4} variable will act through the Θ-function to decrease the P_{B1LR4} variable.

5 Stochastic generation of combat effectiveness

In the initial model we made an extremely simple assumption for the Combat Effectiveness (CE) functions $K_{ab}(M_a, M_b, P_{ab}, P_{ba})$: that they were linear in all variables. Fortunately K_{ab} can be calculated from the average of quick *ab initio* stochastic simulations of skirmishes involving M_a troops on one side and M_b on the other. Here we take the most simple approach assuming that each troop on each side gets one shot at all enemy troops with a probability $p = p_0 P_{ab}$. If we average the difference of the initial troop numbers and those left standing after one round of shooting we end up with a CE function which can be parameterised in M_a,M_b,P_{ab} and P_{ba}.

A test of such a system with $p_0 = 0.3$, $P_{ab} = P_{ba}$ and M_a and M_b ranging from 1 to 6 gives a fitted CE function

$$K_{ab}(M_a, M_b, P_{ab}) = (2 - P_{ab})(0.90 + 0.17M_a)(0.40 - 0.04M_b)P_{ab}M_aM_b \qquad (21)$$

which can be substituted into the ODEs.

In general, care needs to be taken with time-scales and general behaviour. We could take the time-scales into account by a simple premultiplying factor which can be adjusted to simulate different levels of combat strength for these basic runs. However, this should *not* be taken as a viable technique for real simulations.

6 Can the model deal with complex sensitivities?

When we run focussed stochastic analysis to assess the combat effectiveness function on parameters M_a and M_b, we discover a region of high gradient within the CE function. We would need to produce more fine-grained runs around the sharper areas to produce appropriate (nonsmooth) representations of K_{ab}. An example of what such a CE function might look like is shown in Figure 7.

7 Making a stochastic calculation

It seems reasonable to assume that the greatest effect of randomness will be in the outcome of battle situations. Particularly in cases where K_{ab} is highly sensitive to

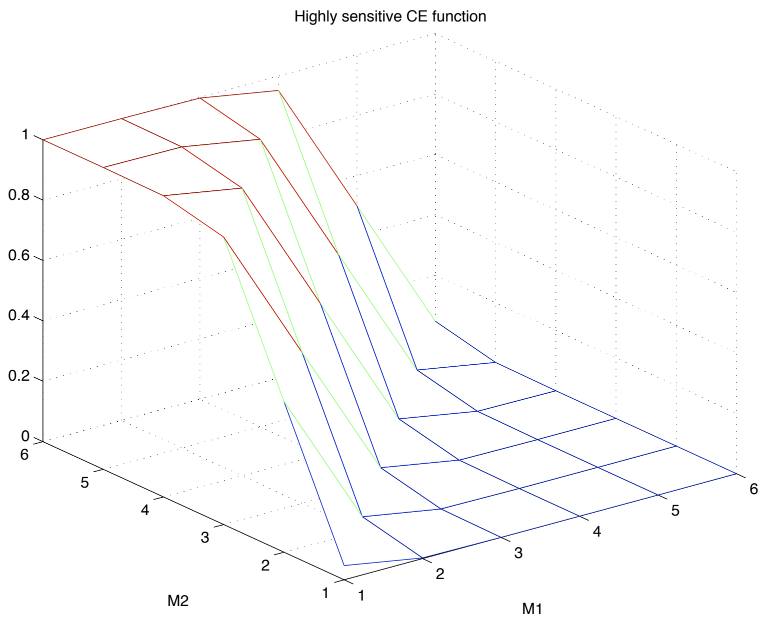

Figure 7: Example of a sensitive Combat Effectiveness function.

variations in M_a or M_b in some region, the global effects can vary strongly on minor variations.

One possible way to model this is through the use of stochastic *effective* M_a, M_b and P_{ab} in the function itself. As an example we would apply

$$\dot{M}_a = -K_{ab}(\tilde{M}_a, \tilde{M}_b, \tilde{P}_{ab}) \tag{22}$$

where

$$\tilde{M}_a = M_a(1 + v_{M_a}\xi_{M_a}) \tag{23}$$
$$\tilde{M}_b = M_a(1 + v_{M_a}\xi_{M_b}) \tag{24}$$
$$\tilde{P}_{ab} = P_{ab}(1 + v_{P_{ab}}\xi_{P_{ab}}) \tag{25}$$

and where the ξs are random values taken from a normal distribution. We have tested this case for our simulations and found good robustness in the evolution of the variables.

8 Summary

We have developed a method for building a combat simulation from elements that can be aggregated into larger simulations. The central idea is to define units of the battle ('effective zones') whose state can be specified through a collection of variables. These variables, which can be continuous or discrete, and can correspond both to physical quantities like number of troops or to nonphysical quantities like combat effectiveness, then evolve governed by a system of coupled (typically nonlinear) ordinary differential equations.

Thus, instead of performing a real time 'turn-by-turn' simulation, a coarse-grain model is set to evolve much like a chemical reaction. An obvious disadvantage is the loss of detailed information about the evolution of small-scale parts of the battle. On the other hand, a coarse-grained simulation of effective zones will evolve exponentially fast to steady-state outcomes.

In this report we have tested the concept of effective zones on a 'standard' battle scenario, and demonstrated both that realistic features emerge from the model and that the simulation is rapid. We have further demonstrated that models of this nature can be aggregated into larger units, or be made more detailed by including and coupling additional variables. Finally we have shown than one can mix elements of stochastic behaviour into the model to make it more realistic.

Modelling combat situations with aggregation via differential equations offers a number of advantages. It is quick and can be shown to fit results from simulation models. By isolating the parameters of interest, it is possible to arrive at a better understanding of the parameter space. A good modelling strategy would use simulation models alongside more mathematical models discussed here since this plays to the strengths of both approaches.