Chapter 6

Monte Carlo Simulation in the Integrated Market and Credit Portfolio Model

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6.1   Introduction

Credit granting institutions deal with large portfolios of assets. These assets represent credit granted to obligors as well as investments in securities. A common size for such a portfolio lies from anywhere between 400 to 10,000 instruments.

The essential goal of the credit institution is to minimize their losses due to default. By default we mean any event causing an asset to stop producing income. This can be the closure of a stock as well as the inability of an obligor to pay their debt, or even an obligor's decision to pay out all his debt.

Minimizing the combined losses of a credit portfolio is not a deterministic problem with one clean solution. The large number of factors influencing each obligor, different market sectors, their interactions and trends, etc. are more commonly dealt with in terms of statistical measures.

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Such include the expectation of return and the volatility of each asset associated with a given time horizon.

In this sense, we consider in the following the expected loss and risk associated with the assets in a credit portfolio over a given time horizon of (typically) 10 to 30 years. We use a Monte Carlo approach to simulate the loss of a portfolio in multiple scenarios, which leads to a distribution function for the expected loss of the portfolio over that time horizon. Second, we compare the results of the simulation to a Gaussian approximation obtained via the Lindeberg-Feller Theorem. Consistent with our expectations, the Gaussian approximation compares well with a Monte Carlo simulation in case of a portfolio of very risky assets.

Using a model which produces a distribution of expected losses allows credit institutions to estimate their maximum expected loss with a certain confidence interval. This in turn helps in making important decisions about whether to grant credit to an obligor, to exercise options or otherwise take advantage of sophisticated securities to minimize losses. Ultimately, this leads to the process of credit risk management.

6.2   The Problem

Estimation of the risk involved in large portfolios of securities posing various individual credit risks is a problem which can be studied using Monte Carlo methods. The main difficulties include

- the large number of different risk factors (interest rates, foreign exchange rates, ...)

- statistical dependencies between market risk factors and probabilities of default.

There are several variance reduction techniques (importance sampling, stratified sampling, ...) which are applicable to many practical problems in finance, in particular to the pricing of sophisticated securities. The problem we face is how to utilize these techniques for portfolio risk analysis.

In general, the problem can be considered in both one-time-step and multi-time-step settings.

The most interesting practical case corresponds to non-risky credit portfolios. In this case the portfolio losses depend on default events that are relatively rare. Therefore, efficient Monte Carlo simulation could be based on a transformation of the measure describing the joint evolution of market and credit risk factors.

A framework for credit risk estimation that has been used in industry is based on a joint market credit risk model described in Idcoe, Kreinin and Rosen [11].

6.3   The Model

We consider a portfolio of individual investments, from which we swap bonds/stocks/derivatives, borrow money, or lend to obligors. Each item (counter party) in our portfolio of investments has a risk of default in form of a probability associated with it. Various individual credit risks are determined by, for example, Moody's ratings and like instruments, plus our own evaluations.
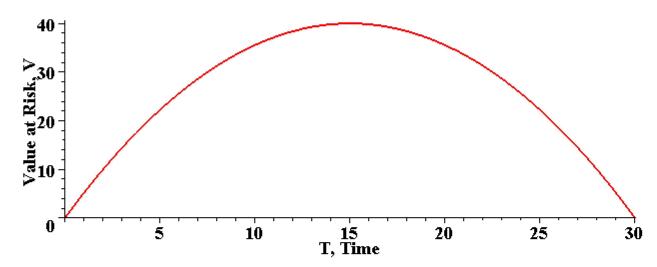
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There can be anywhere from 400 to 10,000 items in a portfolio, from 8 to 200 credit classes which form a partition of the items in the portfolio, and several (independent) risk factors.

Default, again, means the event of an obligor deciding to not pay, or to completely pay out a debt, since in either case our income stops. Stocks, bonds and derivatives also represent obligors in the sense that if the value of a stock associated with an obligor drops to zero, then the obligor has decided not to pay us a return on our investment. In case of default, each item has a corresponding expected loss called the "value at risk" or "exposure", V, of that item.

Figure 1 shows a possible time plot of the exposure of one asset over a 30-year time interval in millions of dollars. The maximum exposure is different for each counter party (obligor/asset), and the time course shown here is typical for a bond swap, which is most common in a credit portfolio.

Unlike most currently used Portfolio Credit Risk (PCR) models, we will assume that market risk factors (interest rates, foreign exchange rates. etc.) are stochastic rather than deterministic. This level of generality is not of particular importance for portfolios of loans and floating rate instruments, but is of great importance for derivatives such as swaps and options.

Our main assumption is that conditional on a market, all defaults and rating changes are independent. The state of our model at any time is a complete specification of the relevant economic and financial credit drivers and market factors (macroeconomic, microeconomic, financial, industrial, etc.) that drive the model.

The actual loss experienced in case of default at time t is described by a random variable, $L = V(t)B$, where $V(t)$ is the exposure at time t, and B is Bernoulli distributed with mean p, the probability of default. The idea is to find the probability of default, p, and obtain the expected loss at time t, $E(L) = pV(t)$.

The framework of our model can be broken down into three parts: Risk Factors and Scenarios, the Joint Default Model, and the Modeling of Obligor Exposures and Recoveries within a Scenario. The Joint Default Model in turn has its own three components: the definitions of Unconditional Default Probabilities, of Credit Worthiness Indices, and the construction of a model which links each obligor's credit worthiness index to the credit drivers. We discuss these parts in detail below.

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6.3.1   Risk Factors and Scenarios

With t measured in years, consider the single period $[t_0, t]$ where $t = t_0 + 1$. In this single time step, a scenario corresponds to one possible state of the world at time t. More precisely, a scenario is identified with a list of K systemic factors which are one possible realization of the corresponding *credit drivers*. The credit drivers in turn are the stochastic quantities which underlie each scenario, and which directly influence the credit worthiness of each obligor in the portfolio. It is common to consider anywhere from 100 to 10,000 possible scenarios.

Let $x(t)$ denote the vector of logarithms of relative risk factors at time t, i.e.,

$$x_k(t) = \ln \left[\frac{r_k(t)}{r_k(t_0)} \right],$$

where $r_k(t)$ denotes the value of the kth risk factor at time t. We assume that at the time horizon the $x_k(t)$ are normally distributed: $\mathbf{x}(t) \sim N(\mu, \mathbf{Q})$, where μ is a vector of mean returns and $\mathbf{Q}$ is a covariance matrix. Denote by $\mathbf{Z}(t)$ the vector after normalization after its components, $Z_k(t) = (x_k(t) - \mu_k) \sigma_k$.

In a single time step, each risk factor $Z_k(t) \sim N(0, 1)$. However, the risk factors are correlated according to $\mathbf{Z}(t) \sim \mathbf{N}(\mathbf{0}, \mathbf{Q})$ with a correlation matrix, $\mathbf{Q}$, which has ones on its diagonal. More precisely, for some empirically obtained number $\rho < 1$

$$\mathbf{Q} = \begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^K \\ \rho & 1 & \rho & \ddots & \vdots \\ \rho^2 & \rho & 1 & \ddots & \rho^2 \\ \vdots & \ddots & \ddots & \ddots & \rho \\ \rho^K & \cdots & \rho^2 & \rho & 1 \end{pmatrix}$$

This correlation structure can be created by starting with a vector of standard normal distributed elements, $\eta \sim \mathbf{N}(\mathbf{0}, \mathbf{I})$, where $\mathbf{I}$ denotes the K-dimensional identity matrix. For some matrix, $\mathbf{A}$, $\mathbf{A}\eta \sim \mathbf{N}(\mathbf{0}, \mathbf{A}\mathbf{A}^\top)$. From a Cholesky factorization for positive definite matrices we obtain $\mathbf{Q} = \mathbf{R}^\top \mathbf{R}$. Therefore, define $\mathbf{A} = \mathbf{R}^\top$ and obtain the correlated risk factors

$$\mathbf{Z}(t) = \mathbf{A}\eta \sim \mathbf{N}(\mathbf{0}, \mathbf{Q})$$

For multiple time steps, the default risk accumulates over time. Therefore we model the risk factors as starting with $\mathbf{Z}(0) = \mathbf{0}$, and evolving according to

$$\mathbf{Z}(t) = \mathbf{Z}(t - 1) + \mathbf{z}$$

where $\mathbf{z}$ is determined by evaluating $\mathbf{Z}(t)$ over a single time step.

6.3.2   The Joint Default Model

The joint default model consists of three components: First, the definition of unconditional default probabilities; second, a multi-factor model of the credit worthiness index for each counter party based upon unconditional default probabilities and credit drivers; and third, an estimate for the conditional default probability of each counter party according to the multi-factor model.

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Unconditional Default Probabilities

Let τ_j denote the time of default by an obligor in sector (credit class) j, and let $p_j(t)$ denote this obligor's *unconditional probability of default*, i.e. the probability of default by an obligor in sector j by time t.

$$p_j(t) = Pr\{\tau_j \leq t\}.$$

We assume that all obligors in sector j have the same unconditional probability of default. We assume that the unconditional probabilities for each sector are available from an internal model or from an external agency. In particular,

$$p_j(t) = 1 - \exp(-\lambda_j t)$$

where the parameters λ are specific to the various credit classes. We use $\lambda_{AAA}^{-1} = 150$ years, $\lambda_D^{-1} = 3$ years, and distribute the values for the remaining six classes linearly in between these values. Here, according to Moody's, AAA is the most reliable credit class, and D is the class of most risky assets.

Credit Worthiness Index

The *credit worthiness index*, $Y_j(t)$, of obligor j, for j=1,...,N, determines the credit worthiness or financial health of counter party j at time t. By considering the value of its index, it can be determined whether an obligor is in default. We define the credit worthiness index by assuming that $Y_j(t)$, a continuous random variable, is related to the credit drivers through a linear multi-factor model as follows. Recall that the number of risk factors is K, therefore

$$Y_j(t) = \sum_{k=1}^{K} \beta_{c(j)k} Z_k(t) + \sigma_{c(j)} \epsilon_j, \tag{cwi}$$

where

$$\sigma_{c(j)} = \left[1 - \sum_{k=1}^{K} \beta_{c(j)k}^2 \right]^{\frac{1}{2}}.$$

is the volatility of the idiosyncratic component associated with the credit class, c(j), of obligor j, and ϵ_j, $j = 1, 2, \ldots, N$ are i.i.d. standard normal variables representing random events affecting obligor j. The coefficients $\beta_{c(j)k}$ correspond to the sensitivity of the index of an obligor in credit class c(j) to the kth risk factor. Therefore, the first term on the right in (1) is the systemic component of the index, while the second term is the idiosyncratic component, specific to each counter party. Note that the distribution of the index is standard normal; it has zero mean and unit variance, which will later allow us to obtain our Gaussian approximation.

Since all obligors in a sector are statistically identical, obligors in a given sector share the same multi-factor model. However, while all obligors in a sector, c, share the same β_{ck}'s and σ_c, each has its own idiosyncratic uncorrelated component ϵ_j.

For implementation purposes we link each credit class, c, to exactly one credit driver, $Z_{k(c)}(t)$, and obtain an index for each obligor j in credit class c:

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$$Y_j(t) = \beta_c Z_{k(c)}(t) + \sigma_c \epsilon_j, \qquad \text{where} \ \ \sigma_c = \left[1 - \beta_c^2\right]^{\frac{1}{2}} \tag{6.1}$$

Values of σ_c^2 are chosen as 0.25, 0.35, 0.55, and 0.8 for the drivers corresponding to the least risky to most risky counter parties, respectively.

The notion of "multi-factor" usually indicates that the credit worthiness index is some non-trivial combination of multiple, independent risk factors. Here, we associate each of our eight Moody type credit classes with one of four risk factors. Therefore, in our framework multi-factor is to be understood in the sense that each driver already is defined as a combination of a set of independent drivers via the correlation matrix. Therefore, it represents the influence of multiple, independent drivers, even though we explicitly include only one risk factor in each credit worthiness index.

Conditional Default Probabilities

The *conditional probability of default* of an obligor in sector j, $p_j(t, \mathbf{Z})$, is the probability that an obligor in sector j defaults at time t, conditional on scenario $\mathbf{Z}$:

$$p_j(t, \mathbf{Z}) = Pr\left\{\tau_j \leq t \mid \mathbf{Z}\right\} \tag{6.2}$$

To estimate these conditional probabilities we will need a conditional default model which describes the functional relationship between the credit worthiness index $Y_j(t)$ (and hence the systemic factors) and the default probabilities $p_j(t, \mathbf{Z})$.

We assume that default is driven by a Merton model [14], i.e., default occurs when the assets of the firm fall below a given boundary or threshold, generally given by its liabilities. In our model the obligor defaults when its credit worthiness index, Y_j, falls below α_j, the *unconditional default threshold*.

In these terms, the unconditional default probability of obligor j is given by

$$p_j = Pr\left\{\tau_j \leq t\right\} = Pr\left\{Y_j < \alpha_j\right\} = \Phi(\alpha_j), \tag{6.3}$$

where Φ denotes the normal cumulative density function (for simplicity we have dropped the time from the notation). Thus the unconditional threshold α_j is obtained from the inverse of equation 6.3. In particular, for a single time step:

$$\alpha_j = \Phi^{-1}(p_j) = \Phi^{-1}(1 - \exp(-\lambda_j t)). \tag{6.4}$$

The conditional probability of default on the other hand is the probability for the credit worthiness index to fall below its threshold in a given scenario. For convenience, we again drop time from the notation:

$$p_j(\mathbf{Z}) = Pr\left\{Y_j < \alpha_j \mid \mathbf{Z}\right\} =$$

$$= Pr\left\{\sum_{k=1}^{q^c} \beta_{jk}\mathbf{Z}_k + \sigma_j \epsilon_j < \alpha_j \mid \mathbf{Z}\right\} = Pr\left\{\epsilon_j < \frac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk}\mathbf{Z}_k}{\sigma_j}\right\} =$$

$$= \Phi\left(\frac{\alpha_j - \sum_{k=1}^{q^c} \beta_{jk}\mathbf{Z}_k}{\sigma_j}\right) = \Phi(\hat{\alpha}_j(\mathbf{Z})).$$

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The *conditional threshold*, $\hat{\alpha}(\mathbf{Z})$, is the threshold below which the idiosyncratic component of obligor j, ϵ_j, must fall for default to occur in a given scenario, $\mathbf{Z}$.

Note that the obligor credit worthiness index correlations are uniquely determined by the default model and the multi-factor model, which links the index to the credit drivers. The correlations between individual obligor defaults are then obtained from the functional relationship between the index and the event of default, as determined by the Merton model. For example, the indices of obligors that belong to the same sector are perfectly correlated if their idiosyncratic component is zero.

6.3.3 Obligor Exposures and Recoveries in a Scenario

We define the exposure to an obligor j at time t as the amount that will be lost due to outstanding transactions with that obligor if default occurs, unadjusted for future recoveries, and we denote it by $V_j(t)$. An important property of PCR-SD is the assumption that exposure is deterministic, not scenario dependent, i.e. $V_j(t)$ is not a function of $\mathbf{Z}$.

The actual loss experienced in case of default of counter party j at time t is described by a random variable,

$$L_j(\mathbf{Z}) = V_j \mathbf{1}$$

where V_j is the exposure at time t, and $\mathbf{1}$ is Bernoulli distributed with mean $p_j(\mathbf{Z})$, the probability of default at time t. The idea is to find the probabilities of default of each counter party and obtain the expected cumulative loss at time t,

$$\mathcal{E}L_j(\mathbf{Z}) = \sum_{j=1}^{N} V_j(t) \cdot p_j(\mathbf{Z}).$$

With adjustment for further recovery, the economic loss for a default by obligor j is

$$L_j(\mathbf{Z}) = V_j \cdot (1 - \gamma_j), \tag{6.5}$$

where γ_j is the recovery rate, expressed as a fraction of the obligor exposure. Recovery in the event of default is assumed deterministic. Expressing the recovery amount as a fraction of the exposure value at default does not necessarily imply instantaneous recovery of that fraction of the exposure at the time of default.

The distribution of conditional losses for each obligor is given by

$$L_j(\mathbf{Z}) = \begin{cases} V_j \cdot (1 - \gamma_j) & \text{with probability } p_j(\mathbf{Z}); \\ 0 & \text{with probability } 1 - p_j(\mathbf{Z}). \end{cases} \tag{6.6}$$

or in short,

$$L_j(\mathbf{Z}) = V_j \cdot (1 - \gamma_j) \cdot \mathbf{1}$$

Our expected loss for a given scenario is given by the sum of the expected losses of each of the obligors:

$$\mathcal{E}L(\mathbf{Z}) = \sum_{j=1}^{N} V_j \cdot (1 - \gamma_j) \cdot p_j(\mathbf{Z}). \tag{6.7}$$

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6.4   Investigation of the Model

In the remainder of this article we will use the following index system: j=1,...,M scenarios, k=1,...,K risk factors, and i=1,...,N default probabilities:

Scenario	Risk Factor	Default Probabilities	Exposures
1	$R_1^1, \ldots, R_K^1$	$P_1^1, \quad P_2^1 \ldots, P_N^1$	$V_1^1, \ldots, V_N^1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
M	$R_1^M, \ldots, R_K^M$	$P_1^M, P_2^M \ldots, P_N^M$	$V_1^M, \ldots, V_N^M$

Let the loss in scenario j be

$$\mathcal{L}_j = \sum_{i=1}^{N} V_i^j \mathbf{1}\{\tau_i^j < \hat{t}\,\}(1 - \gamma_{ji}).$$

This sum has a huge number of terms. Note that

$$\mathcal{E}(\mathbf{1}\{\tau_i^j < \hat{t}\}) = P_i^j.$$

Our key idea is to approximate the distribution.

Phase 1. We use one time interval, $[0, \tau]$, and recovery rates are constant (γ_{ij}).

Phase 2: $1 - \gamma_{ij}$ is random, we still use one time step.

$$Pr\{\gamma_{ij} < t\} = \frac{\int_0^t (1-u)^{\alpha_{ij}} u^{\beta_{ij}} \, du}{\mathcal{B}(\alpha_{ij}, \beta_{ij})}.$$

Phase 3. Multiple time steps.

Assume that γ_{ij} is independent of V_{ij}, at least in the early scenarios. Assume that we know the distribution of the τ_{ij}, that is, $Pr\{\tau_{ij} < t\}$ is known for *all* scenarios and all counterparties j = 1, ..., N, i = 1, ..., k.

We use discrete time and a finite number of steps.

Now think of $R_i^j = R_i^j(t)$ generating $P_i^j(t)$ and $V_i^j(t)$.

How do we develop a distribution for portfolio losses = $\mathcal{L}$? Since you cannot default twice, we need to pull defaults out of $\mathcal{L}$ at each time step:

$$\mathcal{L}(t) = \sum_{k=1}^{t} \mathcal{L}(k).$$

We try

$$\mathcal{L}_j(t) = \sum_{i=1}^{k} V_i^j(t) \mathbf{1}(t - 1 < \tau_i^j \leq t).$$

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Here we have embedded the $1 - \gamma_{ij}(t)$ into the $V_i^j(t)$. Notice that

$$Pr\{t - 1 < \tau_i^j \leq t\} = P_i^j(t) - P_i^j(t - 1).$$

We know the random variables τ_i^j. We would like to use some form of the central limit theorem on this sum.

We assume $\mathcal{L}(t)$ takes on the values $\mathcal{L}_1(t), \ldots, \mathcal{L}_N(t)$ with equal probabilities $1/N$ (uniform distribution).

Typically, the maximum time is in the range 30 to 50.

The problem: How to efficiently estimate $\mathcal{L}_j(t)$, j = 1, $\ldots$, N. We want the distribution of these random variables... moments are not enough. We can write

$$\mathcal{L}_j(T) = \sum_{i=1}^{k} \sum_{t=1}^{T} V_i^j(t) \mathbf{1}\{t - 1 < \tau_i^j \leq t\}.$$

The inner sum in t is denoted $X_i^j(T)$ and represents counterparty loss. These random variables are conditionally independent, given the jth scenario.

Now k is large, say 200-300, so we can use the central limit theorem for triangular arrays, commonly referred to as the Lindeberg- Feller Theorem (see [9]). In order to apply this theorem, we must first center and standardize each of the random variables. Hence we need the mean and variance of the $X_i^j(T)$.

We have:

$$X_i^j(t) = \sum_{t=1}^{T} V_i^j(t) \mathbf{1}\{t - 1 < \tau_i^j \leq t\},$$

$$\mu_i^j(T) = \mathcal{E}X_i^j(T) = \sum_{i=1}^{T} V_i^j(t)[P_i^j(t) - P_i^j(t - 1)].$$

Now to the variance:

$$Var[X_i^j(T) = Var\left[\sum_{t=1}^{T} V_i^j(t) \mathbf{1}\{t - 1 < \tau_i^j \leq t\}\right] =$$

$$= \sum_{i=1}^{T} [V_i^j(t)]^2 Var\{\mathbf{1}\{t - 1 < \tau_i^j \leq t\}\} -$$

$$- \sum_{t=1}^{T} \sum_{\substack{t'=1 \\ t \neq t'}}^{T} V_i^j(t) V_i^j(t') Covar[\mathbf{1}\{t - 1 < \tau_i^j \leq t\} \mathbf{1}\{t' - 1 < \tau_i^j \leq t'\}].$$

Now

$$Covar[\mathbf{1}\{t - 1 < \tau_i^j \leq t\} \mathbf{1}\{t' - 1 < \tau_i^j \leq t'\}] =$$

$$= \mathcal{E}[\mathbf{1}\{t - 1 < \tau_i^j \leq t\} \mathbf{1}\{t' - 1 < \tau_i^j \leq t'\}] - \mathcal{E}[\mathbf{1}\{t - 1 < \tau_i^j \leq t\}] \mathcal{E}[\mathbf{1}\{t' - 1 < \tau_i^j \leq t'\}]$$

The expected value of $[\mathbf{1}\{t - 1 < \tau_i^j \leq t\}]^2$ is

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$$\int [\mathbf{1}\{t-1 < \tau_i^j \leq t\}]^2 \, dP = P_i^j(t) - P_i^j(t-1).$$

while the expected value squared of $\mathbf{1}\{t-1 < \tau_i^j \leq t\}$ is

$$\left[\int \mathbf{1}\{t-1 < \tau_i^j \leq t\} \, dP \right]^2 = [P_i^j(t) - P_i^j(t-1)]^2.$$

Hence

$$Var[\mathbf{1}\{t-1 < \tau_i^j \leq t\}] = P_i^j(t) - P_i^j(t-1) - (P_i^j(t) - P_i^j(t-1))^2.$$

Next we need to calculate

$$\mathcal{E}[\mathbf{1}\{t-1 < \tau_i^j \leq t\}\mathbf{1}\{t'-1 < \tau_i^j \leq t'\}] = \int \mathbf{1}\{t-1 < \tau_i^j \leq t\}\mathbf{1}\{t'-1 < \tau_i^j \leq t'\} \, dP.$$

but t and t' are natural numbers, so for $t \neq t'$ we have $(t-1, t] \cap (t'-1, t'] = \emptyset$, so

$$\mathcal{E}[\mathbf{1}\{t-1 < \tau_i^j \leq t\}\mathbf{1}\{t'-1 < \tau_i^j \leq t'\}] = \int \emptyset \, dP = 0.$$

Finally, we calculate

$$\mathcal{E}[\mathbf{1}\{t-1 < \tau_i^j \leq t\}]\mathcal{E}[\mathbf{1}\{t'-1 < \tau_i^j \leq t'\}] = [P_i^j(t) - P_i^j(t-1)][P_i^j(t') - P_i^j(t'-1)].$$

Putting all of the above together, we have

$$Covar[\mathbf{1}\{t-1 < \tau_i^j \leq t\}\mathbf{1}\{t'-1 < \tau_i^j \leq t'\} = [P_i^j(t) - P_i^j(t-1)][P_i^j(t') - P_i^j(t'-1)].$$

Thus

$$[\sigma_i^j(t)]^2 = Var[X_i^j(T)] = Var\left[\sum_{t=1}^{T} V_i^j(t)\mathbf{1}\{t-1 < \tau_i^j \leq t\} \right] =$$

$$= \sum_{i=1}^{T} [V_i^j(t)]^2 Var\{\mathbf{1}\{t-1 < \tau_i^j \leq t\}\} -$$

$$- \sum_{t=1}^{T} \sum_{\substack{t'=1 \\ t \neq t'}}^{T} V_i^j(t)V_i^j(t')Covar[\mathbf{1}\{t-1 < \tau_i^j \leq t\}\mathbf{1}\{t'-1 < \tau_i^j \leq t'\}] =$$

$$= \sum_{t=1}^{T} ([V_i^j(t)]^2 [P_i^j(t) - P_i^j(t-1)][1 - (P_i^j(t) - P_i^j(t-1))] -$$

$$- [P_i^j(t) - P_i^j(t-1)] \sum_{\substack{t'=1 \\ t \neq t'}}^{T} V_i^j(t)V_i^j(t')[P_i^j(t') - P_i^j(t'-1)]). \tag{6.8}$$

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Let $\mu_i^j(T) = \mathcal{E}X_i^j(T) = \sum_{t=1}^{T} V_i^j(t)[P_i^j(t) - P_i^j(t-1),]$ and let $[\sigma_i^j(t)]^2$ be as above, then define

$$S_k(T) = Var\left(\sum_{i=1}^{k} X_i^j(T)\right) = \sum_{i=1}^{k}\sum_{t=1}^{T} \left[\sigma_i^j(t)\right]^2.$$

Now if we define

$$(*) \qquad Y_{i,k}^j(T) = \frac{X_i^j(T) - \mu_i^j(T)}{\sqrt{S_k(T)}},$$

then we study

$$\sum_{i=1}^{k} Y_{i,k}^j(T).$$

It is easy to see that $\sum_{i=1}^{k} \mathcal{E}\left(Y_{i,k}^j(T)\right)^2 = 1$, which is the first requirement of the Lindeberg-Feller Theorem. The second requirement is that $\forall \epsilon > 0$,

$$\lim_{k\to\infty} \sum_{i=1}^{k} \mathcal{E}\left(|Y_{i,k}^j(T)|^2 \mathbf{1}(|Y_{i,k}^j(T)| > \epsilon)\right) \to 0.$$

If we assume that the V_i^j are bounded, i.e. $\sup_{i,t\leq T} |V_i^j(t)| \leq M^j(T) < \infty$, and that $S_k \to \infty$ as $k \to \infty$, then it is obvious that given $\epsilon > 0$ we have $|Y_{i,k}^j(T)| < \epsilon$ $\forall i \leq k$ for k sufficiently large. Hence the second condition of the Lindeberg-Feller Theorem is easily satisfied, under these very reasonable assumptions. Since the $V_i^j(t)$ represent the loss from counter party i at time t, we are just assuming that our maximum loss up to time T is finite.

Similarly, if we assume the probability of default in a given time interval, $P_i^j(t) - P_i^j(t-1)$. is strictly between 0 and 1 and that $V_i^j(t) > 0$ for $1 \leq t < T$, then we can argue as follows to see that $S_k \to \infty$ as $k \to \infty$. First, note that $S_k(1) < S_k(2) < \cdots < S_k(T)$ since $\sigma_i^j(t) > 0$ $\forall t$. Hence it is enough to consider $S_k(1)$, which is

$$S_k(1) = \sum_{i=1}^{k} \left[\sigma_i^j(1)\right]^2 =$$

$$= \sum_{i=1}^{k} \left[V_i^j(1)\right]^2 \{P_i^j(1) - P_i^j(0)\}\{1 - P_i^j(1) + P_i^j(0)\} \geq$$

$$\geq \sum_{i=1}^{k} [V^j]^2 P^j \quad \to \infty \quad \text{as} \quad k \to \infty,,$$

where $V^j = \inf_i V_i^j(1) > 0$, and $P^j = \inf_i \left[P_i^j(1)(1 - P_i^j(1))\right] > 0$ since both infs are over finite sets of time greater than zero. Note that the double sum in (8) disappears when $T = 1$, since the sum is over the empty set.

So $\mathcal{L}_j(t)$ is approximated by a sum of Gaussian random variables. It is a *mixture* of Gaussian random variables and we know the mean and the variance. So to find the quantile for $\mathcal{L}$ we can

$$\boxed{\pi}$$

use the Central Limit Theorem conditionally, i.e. to find $Pr\left(\mathcal{L} \leq \beta_p\right) = p$ we first condition on the scenario

$$Pr\left(\mathcal{L} \leq \beta_p\right) = \sum_{j=1}^{N} Pr\left(\mathcal{L} \leq \beta_p \mid \text{Scenario } j\right) Pr\left(\text{Scenario } j\right) =$$

$$= \sum_{j=1}^{N} Pr\left(\mathcal{L} \leq \beta_p\right) \frac{1}{N} \approx \sum_{j=1}^{N} \Phi_{\mu_j, \sigma_j}(\beta_p) \frac{1}{N}.$$

Now, for a given p, we can solve numerically for β_p. Since this is an asymptotic result, the only question left is whether k is large enough. This leads to simulation exercises.

6.4.1   Estimating and Simulating Default Probabilities

We have

$$Y(t) = \alpha R(t) + \beta \epsilon(t), \alpha^2 + \beta^2 = 1,$$

with ϵ and R independent. R is the credit driver and indicator of industrial quality, while ϵ is the individual counterpart. If τ is the time of default, then

$$Pr\{\tau = 1\} = Pr\{Y(1) \leq H_1\},$$

$$Pr\{\tau = 2\} = Pr\{Y(1) > H_1, Y(2) \leq H_2\},$$

and so on, where the numbers $H(k)$ represent the boundary of the nondefault region.

$$R(t) = \sum_{1}^{t}[R(j) - R(j-1)],$$

and the differences inside the sum are $N(0, \hat{\sigma})$.

Each counterparty has its own α and β; Y is like an *index*, so we normalize. α can be assumed to lie in the interval $[0.25, 0.8]$. Y(t) is in some sense a probability of default:

$$Pr\{Y(1) < H_1 \mid R(1)\} = Pr\{\epsilon(1) < \frac{H_1 - \alpha R(1)}{\beta}\} =$$

$$= \Phi\left(\frac{H_1 - \alpha R(1)}{\beta}\right)$$

6.4.2   Finding the Unconditional Default Distribution

Define $q_k = Pr\{\tau = k\}, \quad k = 1, 2, \ldots$. These we can determine. We know Y(t) is normal and $Pr\{Y(1) \leq H_1\} = q_1$, so $H_1 = \Phi^{-1}(q_1)$,

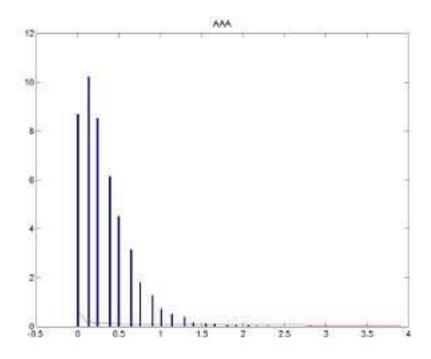$$Pr\{Y(1) > H_1, \ Y(2) < H_2\} = q_2, \quad Y(2) = Y(1) + \Delta Y_2,$$

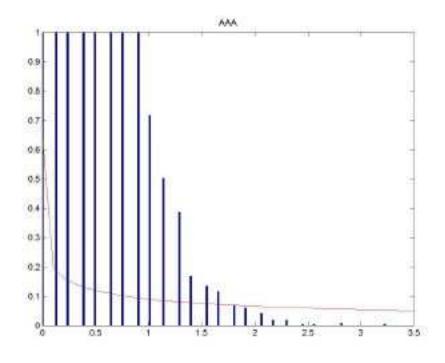and in principle we can carry out all the necessary calculations.
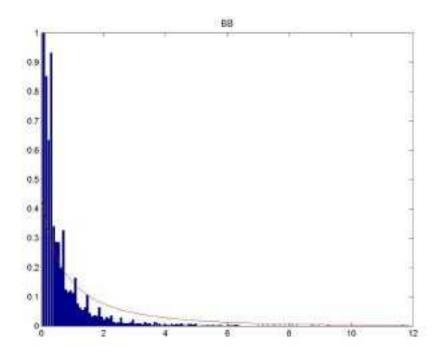
$$\pi$$

6.5 Comparison with Simulations

An alternative to the procedure described above is to use Monte Carlo methods, which in general are much more time-consuming. The effectiveness of our new approach can be judged by running Monte Carlo simulations and comparing with our predictions . In the short time we had, we only were able to run a few simulations, but the results were very promising.

Here are the results of some runs with AAA instruments. The result of the Monte Carlo Simulation is the vertical bars, our new method gives the curve. The figure on the right is a magnification of part of the figure on the left, notice that the best fit is in the "upper tail" :
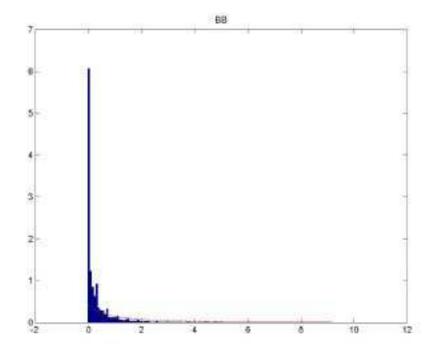
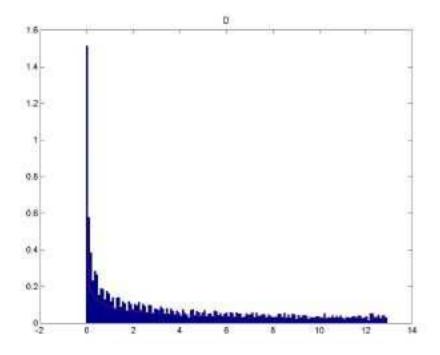Here is the result of a simulation with mixed AAA and BB:

Our model is better with BB instruments:

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and with D instruments:

6.6   Conclusions

By making a clever application of the Lindeberg-Feller Theorem, we were able to develop an analytic method for modelling portfolio losses. The method works well with simulations in the important upper tail of the distributions.

Future work would look at faster algorithms for huge portfolios and try to understand where the Gaussian approximation does and does not work.

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