Temperature Control by Laminar Flows

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1 Model

We consider the following model

$$vT_z = \kappa T_{xx}, \quad 0 < x < 1, \quad 0 < z < 1 + d \tag{1}$$

where T and v are the (non-dimensional) temperature and the speed of the steel bar, respectively. κ is the thermal diffusivity, x and z are the coordinates nondimensionalized by the thickness and the total length of the primary laminar flow cooling region. A secondary (feed-back controlled) flow cooling region is located between the nondimensionalized length $z = 1$ and $z = d$ where $d \ll 1$. Temperature is measured at the inlet ($z = 0$) and exit ($z = 1 + d$) which are denoted by FT (feed temperature) and CT (control temperature), respectively. The boundary conditions are

$$T_x|_{x=0} = 0, \quad T_x|_{x=1} = \begin{cases} -\alpha_L T, & 0 < z < L, \\ 0, & L < z < 1, \\ -\alpha_l T, & 1 < z < 1 + l, \\ 0, & 1 + l < z < 1 + d, \end{cases} \quad T_z|_{z=0} = T_f \tag{2}$$

where α_L, α_l, L and l are the heat transfer coefficients and lengths for the primary and secondary laminar flow cooling regions, respectively and T_f is the temperature at the inlet (FT position).

When v, κ, α_L and α_l are constants, the solution of this initial-boundary-value problem can be obtained using Fourier series. For example, when $L = 1$ and $l = d = 0$, we have

$$T = \sum_n a_n \exp(-\lambda_n^2 z) \cos(\lambda_n \beta x) \tag{3}$$

where $\beta = \sqrt{\kappa/v}$ and λ_n are the eigenvalues determined by

$$\beta \lambda_n \tan(\beta \lambda_n) = \alpha_L.$$

The coefficients a_n are determined by the following equation

$$T_f = \sum_n a_n \cos(\lambda_n \beta x).$$

When $L < 1$ and $d \neq 0$, the exact solution can also be obtained in principle by matching the Fourier expansions for the various regions $0 < z < L$, $L < z < 1$, $1 < z < 1 + l$ and $1 + l < z < 1 + d$.

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2 A Control Problem

To illustrate the basic idea, we consider first the following control problems: Given the inlet temperature T_f, find the correct value of α_L (given v and κ with $L = 1$ and $l = d = 0$) such that the temperature at the surface of bar at the exit ($z = 1 + d$ and $x = 1$) equals to the target temperature T_c^*. In other words, we want to solve the following problem

$$\min_{\alpha_L} F(\alpha_L) \equiv (T_c - T_c^*)^2 \tag{4}$$

where

$$T_c = \sum_n a_n \exp(-\lambda_n^2) \cos(\lambda_n \beta).$$

3 Procedures

In principle, we can consider a Newton type of iteration using the following procedure: we start with an initial guess for the value of $\alpha_L^{(0)}$, we compute the eigenvalues corresponding $\lambda_n^{(0)}$. We can then find the value $T_c^{(0)}$, and using (4), we can find an improvement for α_L using various method. For example, for Newton's method, we have

$$\alpha_L^{(1)} = \alpha_L^{(0)} - \frac{F(\alpha_L^{(0)})}{F'(\alpha_L^{(0)})}.$$

Repeat the process until convergence is achieved, i.e., $\lim_{k \to \infty} |\alpha_L^{(k)} - \alpha_L^{(k-1)}| = 0$ where $\alpha_L^{(k)}$ and $\alpha_L^{(k-1)}$ are the two successive values of the above iteration procedure.

However, this procedure may not converge. Therefore, we use a simpler bisection method in this study since the optimization problem (4) is equivalent to finding the root of $T_c - T_c^* = 0$ when we vary α_L. We start with two initial guesses $\alpha_L^{(1)}$ and $\alpha_L^{(2)}$ and find the approximated value using the recursive formula

$$\alpha_L^{(k)} = \frac{\alpha_L^{(k-1)} + \alpha_L^{(k-2)}}{2}.$$

4 Test Case

To illustrate the procedure, we first considered the each Fourier modes separately due to the linearity of the problem. Another way of looking at it will be to think of T_f is one of the eigenfunctions

$$T_f = \cos(\beta \lambda_n x).$$

The first six eigenvalues computed using Maple for $\beta = 1$ and $\alpha_L = 1$ are 0.8603, 3.4256, 6.4373, 9.5233, 12.6453, 15.7713. The procedure converges for all the eigen-functions.

We then consider the case where the inlet temperature profile is uniform. Since the eigenfunctions are nearly orthogonal, we compute the first six coefficients in the eigenfunction expansion as

$$a_n \approx \frac{\int_0^1 \cos(\beta \lambda_n x) dx}{\int_0^1 \cos(\beta \lambda_n x)^2 dx}$$

and the values are 1.1191, -0.1517, 0.0466, -0.0217, 0.0124, -0.0080.

Simulations have been carried using a Matlab code which does the following:
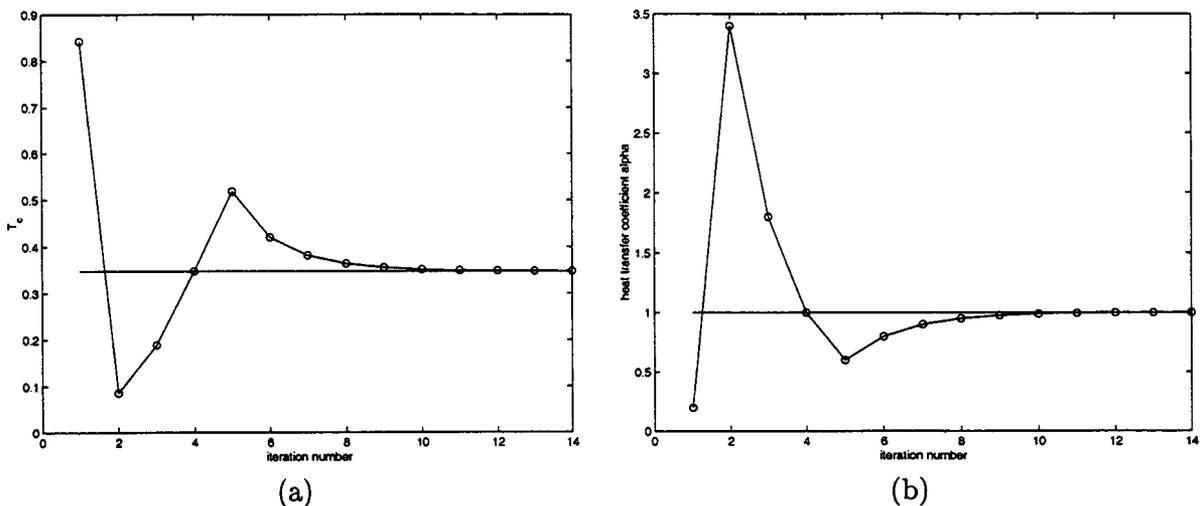
Figure 1: Convergence history for the test case: (a) temperature at the exit T_c; and (b) the heat transfer coefficient α_L. The solid lines are the exact values and circles with line are computational values.

1. Compute the target value $T_c = \sum a_n \exp(-\lambda_n^2) \cos(\lambda_n)$;

2. Pick up two initial values for α_L as $\alpha_L^{(1)} = 0.2$ and $\alpha_L^{(2)} = 3.4$, we can compute the eigenvalues associated with these two values $\lambda_n^{(k)}$, the corresponding exit temperature $T_c^{(k)}$, and the values $F_k = T_c^{(k)} - T_c$, for $k = 1$ and 2;

3. We then use the bisection method to solve the equation $F(\alpha_L) = 0$ with $\alpha_L^{(k)} = (\alpha_L^{(k-1)} + \alpha_L^{(k-2)})/2$ until convergence is reached.

For example, when we set the tolerance $|\alpha_L^{(k)} - \alpha_L^{(k-1)}| < 10^{-3}$, the iteration converges in a few iterations, as shown in Figure 1.

5   Other Control Problems

The case we considered is useful for illustrating the control procedure. It can also be used to find the value of the heat transfer coefficient since α_L is often not known. However, the cases of more practical relevance are to find the length of the laminar cooling regions (l and/or L) so that the target temperature at the exit is achieved. Since the values of physical parameters v and κ may fluctuate, we need to adjust the value of L, more often the length of secondary laminar flow region l based on the measured temperature T_c. In the following, we consider two cases: (1). Controlling primary laminar flow region length L with $l = d = 0$ and $\alpha_L = 1$; (2). Controlling the secondary laminar flow region length l with $L = 0.7$, $\alpha_L = 1$, $d = 0.1$ and $\alpha_l = 0.9$. In both cases, we use the same procedure as the one described earlier with α_L replaced by L or l.

The temperature distributions in the z-direction for the two cases can be obtained by solving the forward problem using a finite difference method described below. Its values on the surface and at the center of the bar are plotted in Figure 2.

5.1   Case 1: controlling primary region ($l = d = 0$)

We now consider the case where α_L is given with no secondary cooling but the length of the primary laminar flow region can be adjusted so that the exit temperature T_c equals to the target
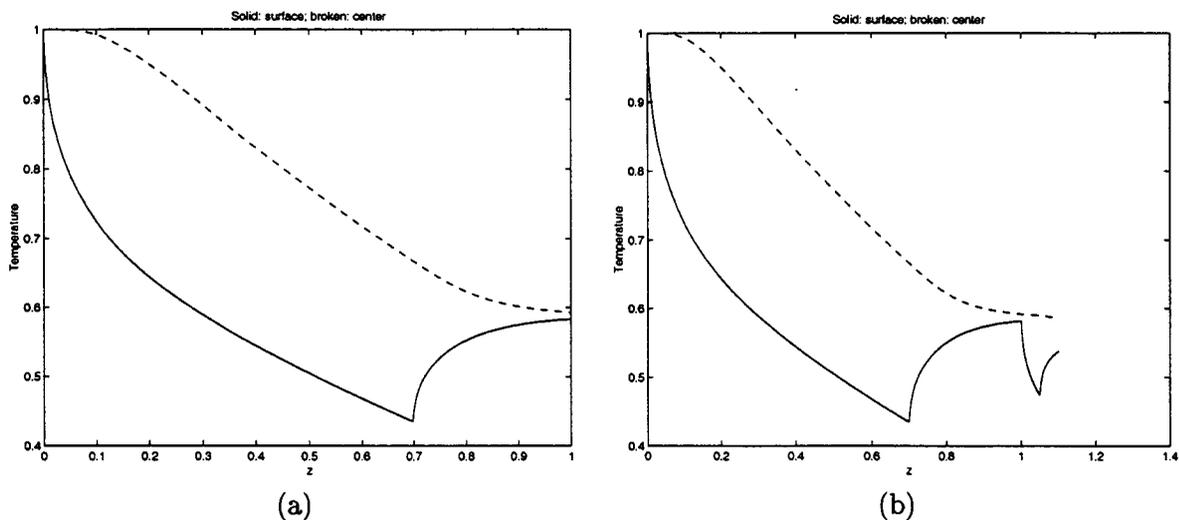
Figure 2: Surface and center temperatures: (a) for Case 1; and (b) for Case 2.

value. In principle, we can use the exact solution represented by the Fourier series in the two regions ($z < L$ and $z > L$). However, it may be more efficient to use numerical method to find an approximated solution for the forward problem.

We use the method of lines to solve the heat equation (1) by discretizing in the x directions only, i.e., replacing T_{xx} by the central difference formula and we obtain

$$\dot{T}_i(z) = \frac{\kappa}{v\delta x^2} \left(T_{i-1}(z) + T_{i+1}(z) - 2T_i(z) \right)$$

for $i = 2, ...N - 1$. For $i = 1$ ($x = 0$) and $i = N$ ($x = 1$), we use the boundary conditions and the equations are

$$\dot{T}_1(z) = \frac{2\kappa}{v\delta x^2} \left(T_2(z) - T_1(z) \right),$$
$$\dot{T}_N(z) = \frac{2\kappa}{v\delta x^2} \left(T_{N-1}(z) - (1 + \alpha_L \delta x)T_N(z) \right)$$

where the $\dot{\{\}}$ denotes the derivative with respect to z. The resulting systems of ordinary differential equations for the grid values T_i and the initial condition is $T_i(0) = T_f$, the inlet temperature.

In Figure 2, we have plotted the convergence history of the control process by setting the objective temperature T_c. Again, we start with two initial guesses, $L = 0$ and $L = 1$, the correct value for the cooling region is found after few iterations.

5.2 Case 2: controlling secondary cooling region

In the manufacturing process, controlling the secondary cooling region length is used more often as a feed-back controlling device in response to possible variations in v (speed of moving steel bar) or κ (heat conductivity due to difference in steel composition, etc.). Here we consider the case where the primary cooling region length is set to be 1 and the heat transfer coefficients $\alpha_L = 1$ and $\alpha_l = 0.9$. We set the target temperature to be the one corresponding to $v = 1$, $\kappa = 1$, $d = 0.1$ and $l = 0.05$. We assume that l is unknown and use the iterative procedure to find its value.

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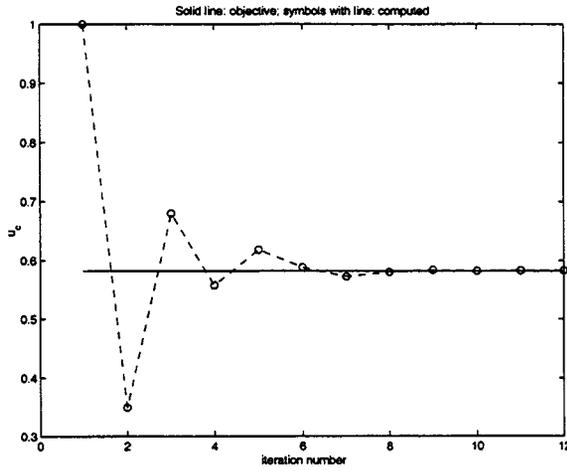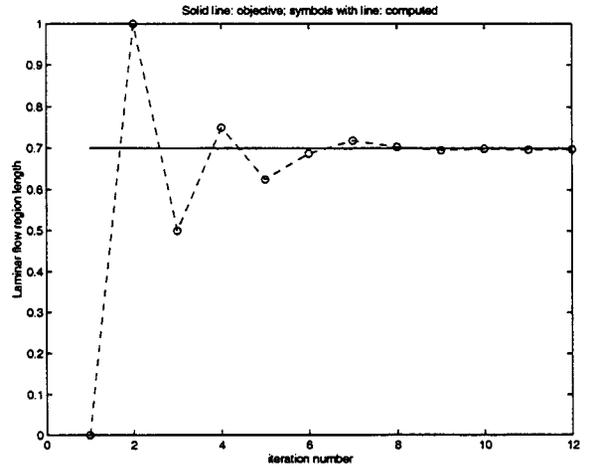
Figure 3: Convergence history for Case 1: (a) for temperature at the exit T_c and (b) for the length of the primary cooling region L. The solid lines are the exact values and circles with line are computational values.
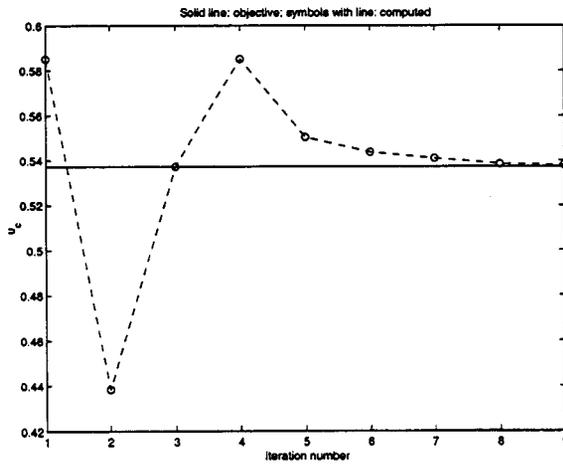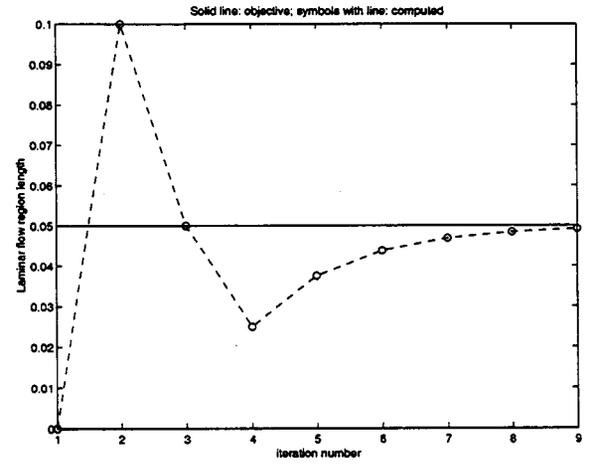
Figure 4: Convergence history for Case 2: (a) for temperature at the exit T_c and (b) for the length of the secondary cooling region l. The solid lines are the exact values and circles with line are computational values.

In Figure 4, we have plotted the convergence history of the control process by setting the objective temperature T_c. Again, we start with two initial guesses, $l = 0$ and $l = d$, corresponds to two cases with no cooling and cooling applied to the entire region. The correct value for the cooling region ($l = 0.05$) is found after few iterations.

6   Discussion

We have demonstrated that given the inlet temperature, the procedure of using the heat transfer coefficient α_L or the cooling region lengths (L or l) to control the **surface** temperature T_c is well-posed and the iterative procedure converges rapidly. An analysis by Dr. John Ockendon, also included in this report, on a slightly simpler problem using Neumann condition $T_x|_{x=1} = -\alpha$ with a primary cooling region showed that the problem is indeed well-posed.