Chapter 4
Approximate solution to a hybrid model with stochastic volatility: a singular-perturbation strategy

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Abstract:
We study a hybrid model of Schöbel-Zhu-Hull-White-type from a singular-perturbation-analysis perspective. The merit of the paper is twofold: On one hand, we find boundary conditions for the deterministic non-linear degenerate parabolic partial differential equation for the evolution of the stock price. On the other hand, we combine two-scales regular- and singular-perturbation techniques to find an approximate solution to the pricing PDE. The aim is to produce an expression that can be evaluated numerically very fast.
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4.1 Introduction

Although the famous Black-Scholes model has been widely applied to price plain vanilla options, comparisons with data analysis of real markets show that some of the assumptions beyond the Black-Scholes equations are unrealistic. It seems that one of the major reasons why this inconsistency happens is the use of the constant volatility modeling assumption. Recently, a lot of attention is paid to more general volatility models - in particular for cases where the volatility is governed by a stochastic differential equation; compare [8] for a brief discussion of these aspects. Very popular in this class of models is the Schöbel-Zhu scenario, where the volatility is driven by a mean-reverting Ornstein-Uhlenbeck process [9, 10]. We refer the reader to [17] for an accessible introduction to the topic of options pricing and to [4, 14], e.g., for a presentation of concepts related to the involved stochastic differential equations.

The problem posed by Rabobank to the 64th *European Study Group Mathematics With Industry* was the following:

(A) Assuming non-zero-correlation between the processes, develop a hybrid model that can handle the stochastic behavior of both the volatility for the equity product and the interest rates.

(B) Use singular-perturbation methods, construct an approximate solution to the linear degenerate partial-differential equation arising in the context of pricing European-style options when the governing asset process is defined by a Schöbel-Zhu-Hull-White hybrid model, which satisfies the requirements mentioned in (A).

This paper is organized in the following fashion: In Section 4.2 we concisely describe the so-called Schöbel-Zhu-Hull-White hybrid model and indicate the form of the partial differential equation (PDE) for pricing an European option. We also mention at this point some of the main theoretical difficulties that this PDE involves. The derivation of the PDE is reported in Section 4.3. Partly based on our "physical" intuition and partly based on the Black-Scholes methodology, we propose boundary conditions for the pricing PDE. The bulk of the paper, that is Section 4.4, contains our singular-perturbation solution strategy. Section 4.5 contains our main result, i.e the approximate expression for the price given by (4.21). We discuss here a few aspects that we consider as relevant when using perturbation approaches to pricing plain-vanilla claims under multi-asset models.

4.2   Problem description

In this note we study the following Schöbel-Zhu-Hull-White hybrid model, viz.

$$
\begin{cases}
\mathrm{d}S_t = & r_t S_t \mathrm{d}t + & \sigma_t S_t \mathrm{d}W_t^S \\
\mathrm{d}\sigma_t = & \kappa(\overline{\sigma} - \sigma_t)\mathrm{d}t + & \eta \mathrm{d}W_t^\sigma \\
\mathrm{d}r_t = & \lambda(\overline{r} - r_t)\mathrm{d}t + & \gamma \mathrm{d}W_t^r
\end{cases}
\tag{4.1}
$$

Here W_t^S, W_t^σ and W_t^r denote standard Brownian motions with quadratic covariation processes $\mathrm{d}W_t^S \mathrm{d}W_t^\sigma = \rho_{S\sigma}\mathrm{d}t$ and likewise for ρ_{Sr} and $\rho_{\sigma r}$. Furthermore, $\rho_{SS} = \rho_{\sigma\sigma} = \rho_{rr} = 1$. We mention that W_t^S, W_t^σ and W_t^r are standard Brownian motions under the risk neutral measure $\mathbb{Q}$. Note that the model given by the first two equations and with constant interest rate, is investigated in [16]. In what follows, we refer to (4.1) as SZHW.

A European call option is a contract that gives the buyer of the contract the right to buy a number of shares from the writer of the contract at a specified time T in the future, the expiry date, for a fixed price K, the strike price of the option. Because, the writer possibly has to sell shares to the option holder for a price less than their value on the stock market the buyer pays a premium to the writer, this is the price of the option at $t = 0$. At expiry the value of the option is $\max(S(T) - K, 0)$ where S is the price of the underlying stock at expiry. The central question in pricing of derivatives is: What is the price of the option at time $0 < t < T$, which is calculated by determining its price at all times between $t = 0$ and expiry?

In Section 4.3, we derive the pricing PDE for an European option

$$
\begin{aligned}
0 = & \ \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{\partial V}{\partial \sigma}\kappa(\overline{\sigma} - \sigma) + \frac{\partial V}{\partial r}\lambda(\overline{r} - r) + \\
& \ \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\eta^2 + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\gamma^2 + \\
& \ \frac{\partial^2 V}{\partial S \partial \sigma}\sigma S \eta \rho_{S\sigma} + \frac{\partial^2 V}{\partial S \partial r}\sigma S \gamma \rho_{Sr} + \frac{\partial^2 V}{\partial \sigma \partial r}\eta\gamma\rho_{\sigma r} - rV.
\end{aligned}
\tag{4.2}
$$

The SZHW model allows σ and r to become negative. When σ is negative, it should be noted that the correlation between changes in time of S and changes in σ reverses in sign. We remark that this causes degeneracies at several places in the pricing PDE. To be more precise, for $\sigma = 0$ the determinant of the diffusion matrix vanishes. We do not treat these difficulties here (see also Remark 4.1), but we suggest three possible solutions:

The first one is the introduction of a positive function $f(\sigma)$. The stochastic differential equation for S is then replaced by $\mathrm{d}S_t = r_t S_t \mathrm{d}t + f(\sigma_t)S_t \mathrm{d}W_t^S$. This approach has been adopted in [5], e.g.

The second solution is the Heston-Cox-Ingersoll-Ross model, see for example [8]. In the SZHW S_t cannot become negative because of the $S\mathrm{d}W$ in the equation. In the

Heston-Cox-Ingersoll-Ross model the potential negativity of σ is removed in a similar way.

A third solution is to take κ large. If κ is large then if σ_t becomes negative it is pushed back very fast towards the value $\bar{\sigma}$. Thus, we might still produce realistic results if we only allow for positive σ in the pricing PDE. We adopt here the third approach.

4.3   Derivation of a deterministic PDE

Consider the SZHW, see (4.1). We define

$$V(t, S_t, \sigma_t, r_t) = B(t)\mathbb{E}^{\mathbb{Q}}\left(\frac{\max(S_T - K, 0)}{B(T)}\,\Big|\,\mathcal{F}_t\right) = \mathbb{E}^{\mathbb{Q}}\left(\frac{\max(S_T - K, 0)}{B(T)/B(t)}\,\Big|\,\mathcal{F}_t\right)$$

. Here $B(t) = \exp\left(\int_0^t r_s \mathrm{d}s\right)$ and $\mathcal{F}_t = \sigma(S_s, \sigma_s, r_s; s \leq t)$. In particular $B(t)$ satisfies the "ordinary" differential equation

$$\mathrm{d}B(t) = r_t B(t)\mathrm{d}t.$$

We are very well aware of the fact that the coefficients in (4.1), in particular the coefficient $\sigma_t S_t$, do not satisfy the usual Lipschitz condition for an Itô diffusion. This might cause difficulties, for example in ensuring the existence of solutions of SZHW model in the precise time interval of interest for the financial situation, cf. [13], for a solution see [8, 9]. In this paper, we waive these complications and *assume* that there exists a differentiable function $\Pi = \Pi(t, S, \sigma, r)$ such that

$$\mathbb{E}^{\mathbb{Q}}\left(\frac{\max(S_T - K, 0)}{B(T)}\,\Big|\,\mathcal{F}_t\right) = \frac{V(t, S_t, \sigma_t, r_t)}{B(t)} = \Pi(t, S_t, \sigma_t, r_t).$$

We postpone the investigation of the existence of Π for a later stage. It is clear from the definition that $\Pi_t = \Pi(t, S_t, \sigma_t, r_t)$ is a martingale. Since $B(t)$ is such a simple process, Itô formula leads to

$$\mathrm{d}\Pi_t = \mathrm{d}\left(\frac{V_t}{B(t)}\right) = \frac{1}{B(t)}\mathrm{d}V_t - r_t \frac{V_t}{B(t)}\mathrm{d}t. \tag{4.3}$$

Now, we derive the Itô differential equation for V. Using Itô formula Theorem 4.2.1 from [14], we obtain

$$
\begin{aligned}
\mathrm{d}V_t \;=\;& \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S_t + \frac{\partial V}{\partial \sigma}\mathrm{d}\sigma_t + \frac{\partial V}{\partial r}\mathrm{d}r_t + \\
& \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\mathrm{d}S_t\mathrm{d}S_t + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\mathrm{d}\sigma_t\mathrm{d}\sigma_t + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\mathrm{d}r_t\mathrm{d}r_t + \\
& \frac{\partial^2 V}{\partial S\partial \sigma}\mathrm{d}S_t\mathrm{d}\sigma_t + \frac{\partial^2 V}{\partial S\partial r}\mathrm{d}S_t\mathrm{d}r_t + \frac{\partial^2 V}{\partial \sigma\partial r}\mathrm{d}\sigma_t\mathrm{d}r_t \\
=\;& \frac{\partial V}{\partial t}\mathrm{d}t + \frac{\partial V}{\partial S}\mathrm{d}S_t + \frac{\partial V}{\partial \sigma}\mathrm{d}\sigma_t + \frac{\partial V}{\partial r}\mathrm{d}r_t + \\
& \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma_t^2 S_t^2\mathrm{d}t + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\eta^2\mathrm{d}t + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\gamma^2\mathrm{d}t + \\
& \frac{\partial^2 V}{\partial S\partial \sigma}\sigma_t S_t\eta\rho_{S\sigma}\mathrm{d}t + \frac{\partial^2 V}{\partial S\partial r}\sigma_t S_t\gamma\rho_{Sr}\mathrm{d}t + \frac{\partial^2 V}{\partial \sigma\partial r}\eta\gamma\rho_{\sigma r}\mathrm{d}t.
\end{aligned}
$$

Eventually, by the martingale representation theorem Theorem 4.3.4 of [14], the $\mathrm{d}t$ term in the full expansion of Eqn. (4.3) in $\mathrm{d}t$, $\mathrm{d}W_t^S$, $\mathrm{d}W_t^\sigma$ and $\mathrm{d}W_t^r$ has to vanish. After multiplication with $B(t)$ it leads to a pricing PDE Eqn. (4.2)

$$
\begin{aligned}
0 \;=\;& \frac{\partial V}{\partial t} + \frac{\partial V}{\partial S}rS + \frac{\partial V}{\partial \sigma}\kappa(\overline{\sigma} - \sigma) + \frac{\partial V}{\partial r}\lambda(\overline{r} - r) + \\
& \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 + \frac{1}{2}\frac{\partial^2 V}{\partial \sigma^2}\eta^2 + \frac{1}{2}\frac{\partial^2 V}{\partial r^2}\gamma^2 + \\
& \frac{\partial^2 V}{\partial S\partial \sigma}\sigma S\eta\rho_{S\sigma} + \frac{\partial^2 V}{\partial S\partial r}\sigma S\gamma\rho_{Sr} + \frac{\partial^2 V}{\partial \sigma\partial r}\eta\gamma\rho_{\sigma r} - rV.
\end{aligned}
$$

We look for a solution V which is bounded by a polynomial in (S, σ, r). The final condition, given at $t = T$, is

$$
V(T, S, \sigma, r) = B(T)\frac{\max(S - K, 0)}{B(T)} = \max(S - K, 0), \tag{4.4}
$$

where K is the strike price of the call option. It is worth noting that the above procedure provides a deterministic PDE for the price evolution but does not specify the boundary conditions needed to close the formulation of the problem. The solution being bounded by a polynomial in its variables may be enough as boundary condition. Based upon the solution and boundary conditions typically used for the Black-Scholes equation as well as by the "physics" of the problem, we suggest the following boundary conditions:

$$
\begin{aligned}
V \;&\to\; 0 \text{ as } r \to -\infty, \tag{4.5}\\
V \;&\sim\; S \text{ as } S \to \infty,\ \sigma \to \infty \text{ or } r \to \infty,\\
V \;&\to\; 0 \text{ as } S \to 0\\
V \;&\sim\; S - Ke^{-r(T-t)} \text{ as } \sigma \to -\infty.
\end{aligned}
$$

This is one of the important results of this paper. Note that depending on the financial scenario in question, other boundary conditions might be employed. The fundamental question which needs to be addressed is: To which extent such choices of boundary conditions lead to well-posed PDEs? We refer the reader to [17] Section 3.7 for a nice and inspiring discussion of the boundary conditions to the Black-Scholes equation.

4.4 Our solution strategy

Our basic idea is to combine regular and singular perturbation techniques to analyze the parabolic PDE for V (arising when pricing the options in the presence of stochastic volatility) for a non-degenerate scenario in the presence of couple of characteristic time scales. The forthcoming sections have the following structure. In Section 4.4.1 we discuss a slightly different model and a reference in which perturbation methods are applied to this model. We believe these results can be extended to the SZHW model. Unfortunately, a full extension of these results is not feasible within the scope of the study group. In the remaining sections we make a step towards extending these results to the SZHW model.

4.4.1 Perturbation methods applied to a slightly different model

In [5] the authors discuss the following model

$$
\begin{cases}
\mathrm{d}X_t = & \mu X_t \mathrm{d}t+ & f\left(Y_t, Z_t\right) X_t \mathrm{d}W_t^X \\
\mathrm{d}Y_t = & \dfrac{1}{\epsilon}(m - Y_t)\mathrm{d}t+ & \dfrac{\nu\sqrt{2}}{\sqrt{\epsilon}}\mathrm{d}W_t^Y \\
\mathrm{d}Z_t = & \delta c\left(Z_t\right)\mathrm{d}t+ & \sqrt{\delta} g\left(Z_t\right)\mathrm{d}W_t^r,
\end{cases}
\tag{4.6}
$$

where both $\epsilon, \delta \ll 1$ and the three stochastic processes are correlated. In this model the stochastic processes for Y and Z should be interpreted as a *fast* and a *slow* volatility. This model differs from the SZHW model in the first equation. In this model the first equation depends on Z (the third equation) through the function f in front of the stochastic term $\mathrm{d}W_t^X$. In the SZHW model the dependence on the third equation appears in front of the deterministic term $\mathrm{d}t$. Apart from only suggesting an asymptotic expansion, the authors of [5] also discuss the error analysis making use of higher order terms in their expansion. Additionally, they also perform a calibration of their solution to existing data. Here we concentrate on finding the asymptotic expansion. To this end, we apply a perturbation method involving two scales to approximate SZHW model in some limiting situations. In Section 4.4.2 we describe the basic setup, in Section 4.4.3 we discuss the limit $\epsilon \to 0$, while in Section 4.4.4 we discuss the second limit $\delta \to 0$. In Section 4.5 we list our expansion.

Note that Section 2.6.2 of the PhD thesis [18] contains a summary of the multiscale expansion developed in [5]. Both [11] and [12] report on a detailed perturbation analysis for the fast mean reverting model (consisting of only the first two equations).

4.4.2 Set-up

Consider the SZHW model (4.1). Analogously to the approach in [5], we look to the scales

$$\kappa = \frac{\bar{\kappa}}{\epsilon}, \ \eta = \frac{\bar{\eta}}{\sqrt{\epsilon}}, \ \lambda = \delta\bar{\lambda}, \ \gamma = \sqrt{\delta}\bar{\gamma}. \tag{4.7}$$

In terms of these scales, the SZHW model becomes

$$\begin{cases} \mathrm{d}S_t = & r_t S_t \mathrm{d}t+ & \sigma_t S_t \mathrm{d}W_t^S \\ \mathrm{d}\sigma_t = & \frac{\bar{\kappa}}{\epsilon}(\bar{\sigma} - \sigma_t)\mathrm{d}t+ & \frac{\bar{\eta}}{\sqrt{\epsilon}}\mathrm{d}W_t^\sigma \\ \mathrm{d}r_t = & \delta\bar{\lambda}(\bar{r} - r_t)\mathrm{d}t+ & \sqrt{\delta}\bar{\gamma}\mathrm{d}W_t^r. \end{cases} \tag{4.8}$$

We note that the second equation can be obtained from the second equation in (4.1) by scaling time with a factor $\frac{1}{\epsilon}$ and that the third can be obtained from the third equation in (4.1) by scaling time with a factor δ. Intuitively the choice of these scales implies that the volatility σ is pushed very fast towards the average value $\bar{\sigma}$. Furthermore, we expect that the interest rate r evolves very slowly in time, and thus is approximately constant on short time scales.

If we set $S = e^x$ and choose only one of the correlations $\rho_{\sigma r}$ to vanish, then according to the derivation in Section 4.3 the corresponding PDE becomes

$$V_t + \frac{\sigma^2}{2}V_{xx} + \frac{\bar{\eta}^2}{2\epsilon}V_{\sigma\sigma} + \frac{\bar{\gamma}^2\delta}{2}V_{rr} + \sigma\frac{\bar{\eta}}{\sqrt{\epsilon}}\rho_{S\sigma}V_{x\sigma} + \sigma\bar{\gamma}\sqrt{\delta}\rho_{Sr}V_{xr} \tag{4.9}$$

$$+ \frac{\bar{\kappa}}{\epsilon}\left(\bar{\sigma} - \sigma\right)V_\sigma + \bar{\lambda}\delta\left(\bar{r} - r\right)V_r + \left(r - \frac{\sigma^2}{2}\right)V_x - rV = 0.$$

The correlation $\rho_{\sigma r}$ is the instantaneous correlation between the short rate process r_t and the volatility process σ_t. In practice this additional parameter could be used as an additional degree of freedom in the calibration. However, for simplicity we set this correlation equal to zero while assuming non-zero correlation between: the stock process S_t and the interest rate process r_t, ρ_{Sr}, and the stock process S_t and the volatility process σ_t, $\rho_{S\sigma}$.

Remark 4.1. Note that if σ vanishes, then some of the "diffusivities" vanish as well, and hence, (4.9) becomes a *degenerate* parabolic equation. Trusting the analysis work by Achdou et al. (see, for instance, [1, 2]) we expect that a variational analysis involving

weighted Sobolev spaces and the theory of semigroups may enable us to prove the existence and uniqueness of weak solutions as well as a maximum principle. From a practical point of view, the role of such an analysis is to yield a unique positive and polynomially bounded price V. It is worth noting that the PDE (4.9) might be also viewed as a diffusion equation for infinite fissured media (somehow in the spirit of [3]). As far as we know, this perspective is rich in new ideas and we think that it deserves further analytical investigation.

To solve this PDE we are going to use both singular and regular perturbation methods for two different small parameters, namely ϵ and δ. We take for granted that the price V can be approximated by an asymptotic expansion in terms of ϵ and δ as

$$V = V_0 + \sqrt{\epsilon}V_1 + \sqrt{\delta}V_2 + O(\delta, \epsilon).$$

In the next two sections we look at the limits $\epsilon \to 0$ and $\delta \to 0$ separately.

4.4.3   The limit $\epsilon \to 0$

We wish now to treat the case ϵ small and compute the terms V_0 and V_2 of the formal expansion of V. In this case the volatility is fluctuating very fast with a fixed variance, and we deduce from [5] Definition 3.3 and [12] equation (22) that the effect of this for the PDE is that we can take constant volatility $\bar{\sigma}$. Thus, using these references we obtain that in the limit $\epsilon \to 0$ the PDE simplifies and takes the form

$$V_t + \frac{\bar{\sigma}^2}{2}V_{xx} + \frac{\bar{\gamma}^2\delta}{2}V_{rr} + \bar{\sigma}\bar{\gamma}\sqrt{\delta}\rho_{Sr}V_{xr} + \bar{\lambda}\delta\left(\bar{r} - r\right)V_r + \left(r - \frac{\bar{\sigma}^2}{2}\right)V_x - rV = 0. \quad (4.10)$$

Note that V_0 does not depend on σ but only on $\bar{\sigma}$. In this way it is intuitively clear that that $O\left(\epsilon^{-1}\right)$ terms in (4.9) vanish, see [12] equation (22) for a detailed discussion of this argument.

Since in the PDE the coefficients in front of the second order derivatives are constant, we can apply the transformation

$$v(x, r, t) = e^{Ax+Br+Ct}V(x, r, t, \epsilon = 0)$$

where A, B, C are functions of (x, r). By means of an appropriate choice of A, B, and C

we obtain an equation without first-order terms. Choosing

$$A = -\frac{1}{2}\frac{2\delta^{3/2}\rho_{Sr}\bar{\lambda}r\bar{\sigma} - 2\delta^{3/2}\rho_{Sr}\bar{\lambda}\bar{r}\bar{\sigma} + 2\gamma r - \gamma\bar{\sigma}^2}{\bar{\sigma}^2\gamma(\delta\rho_{Sr}^2 - 1)},$$

$$B = \frac{1}{2}\frac{2\lambda\delta r\bar{\sigma} + 2\gamma\sqrt{\delta}\rho_{Sr}r - \gamma\sqrt{\delta}\rho_{Sr}\bar{\sigma}^2 - 2\lambda\delta\bar{r}\bar{\sigma}}{\gamma^2(\delta\rho_{Sr}^2 - 1)\bar{\sigma}},$$

$$C = -\frac{1}{4}\frac{1}{\bar{\sigma}^2\gamma^2(\delta\rho_{Sr}^2 - 1)^2}(4\gamma^2 r\bar{\sigma}^2\delta^2\rho_{Sr}^4 - 8\delta\rho_{Sr}^2\gamma^2 r\bar{\sigma}^2 - 12\delta^{3/2}\rho_{Sr}\gamma r\lambda\bar{r}\bar{\sigma} + 4\lambda\delta^{5/2}\bar{r}\bar{\sigma}\gamma\rho_{Sr}^3 r$$

$$- 6\gamma\bar{\sigma}^3\delta^{3/2}\rho_{Sr}\lambda r + 6\gamma\bar{\sigma}^3\delta^(3/2)\rho_{Sr}\lambda\bar{r} + 4\gamma^2 r^2 + \gamma^2\bar{\sigma}^4 + 2\gamma\bar{\sigma}^3\delta^{5/2}\rho_{Sr}^3\lambda r - 4\gamma r^2\delta^{5/2}\rho_{Sr}^3\lambda\bar{\sigma}$$

$$- 2\lambda\delta^{5/2}\bar{r}\bar{\sigma}^3\gamma\rho_{Sr}^3 + 12\delta^{3/2}\rho_{Sr}\gamma r^2\lambda\bar{\sigma} + 4\lambda^2\delta^2 r^2\bar{\sigma}^2 + 4\lambda^2\delta^2\bar{r}^2\bar{\sigma}^2 - 8\lambda^2\delta^2\bar{r}\bar{\sigma}^2 r),$$

we obtain

$$v_t + \frac{1}{2}\sigma^2 v_{xx} + \frac{1}{2}\gamma^2 v_{rr} + \sigma\gamma\rho_{Sr}v_{xr} = 0. \tag{4.11}$$

We eliminate the cross terms with a rotation of the axes given by the transformation

$$\begin{cases} X = \dfrac{\frac{1}{2}\sigma\rho_{Sr}\gamma}{\sqrt{\left(\frac{1}{2}\sigma\rho_{Sr}\gamma\right)^2 + \left(\frac{1}{2}\sigma^2 - \left(\frac{1}{4}\gamma^2 + \frac{1}{4}\sigma^2 + \frac{1}{4}\sqrt{(\gamma^2 + \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)\right)^2}}x \\ \qquad - \dfrac{\frac{1}{2}\sigma\rho_{Sr}\gamma}{\sqrt{\left(\frac{1}{2}\sigma\rho_{Sr}\gamma\right)^2 + \left(\frac{1}{2}\sigma^2 - \left(\frac{1}{4}\gamma^2 + \frac{1}{4}\sigma^2 - \frac{1}{4}\sqrt{(\gamma^2 + \sigma^2) + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)\right)^2}}r \\ R = -\dfrac{\frac{1}{2}\sigma^2 - \left(\frac{1}{4}\gamma^2 + \frac{1}{4}\sigma^2 + \frac{1}{4}\sqrt{(\gamma^2 + \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)}{\sqrt{\left(\frac{1}{2}\sigma\rho_{Sr}\gamma\right)^2 + \left(\frac{1}{2}\sigma^2 - \left(\frac{1}{4}\gamma^2 + \frac{1}{4}\sigma^2 + \frac{1}{4}\sqrt{(\gamma^2 + \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)\right)^2}}x \\ \qquad + \dfrac{\frac{1}{2}\sigma^2 - \left(\frac{1}{4}\gamma^2 + \frac{1}{4}\sigma^2 - \frac{1}{4}\sqrt{(\gamma^2 + \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)}{\sqrt{\left(\frac{1}{2}\sigma\rho_{Sr}\gamma\right)^2 + \left(\frac{1}{2}\sigma^2 - \left(\frac{1}{4}\gamma^2 + \frac{1}{4}\sigma^2 - \frac{1}{4}\sqrt{(\gamma^2 + \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)\right)^2}}r. \end{cases} \tag{4.12}$$

Thus we arrive at an equation of the form

$$v_t + \frac{1}{2}\left(\frac{1}{2}\gamma^2 + \frac{1}{2}\sigma^2 + \frac{1}{2}\sqrt{(\gamma^2 - \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)v_{XX} \tag{4.13}$$

$$+ \frac{1}{2}\left(\frac{1}{2}\gamma^2 + \frac{1}{2}\sigma^2 - \frac{1}{2}\sqrt{(\gamma^2 - \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}\right)v_{RR} = 0,$$

that is

$$v_t + \frac{1}{2}\alpha^2 v_{XX} + \frac{1}{2}\beta^2 v_{RR} = 0, \tag{4.14}$$

where

$$\alpha = \sqrt{\tfrac{1}{2}\gamma^2 + \tfrac{1}{2}\sigma^2 + \tfrac{1}{2}\sqrt{(\gamma^2 - \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}},$$

$$\beta = \sqrt{\tfrac{1}{2}\gamma^2 + \tfrac{1}{2}\sigma^2 - \tfrac{1}{2}\sqrt{(\gamma^2 - \sigma^2)^2 + 4\sigma^2\gamma^2\rho_{Sr}^2}}.$$

After performing all these transformations we derived the backward heat equation from equation (4.10). By introducing a new change of variables

$$\tau = T - t, \ \hat{x} = \frac{X}{\alpha}, \ \hat{r} = \frac{R}{\beta} \tag{4.15}$$

we finally obtain

$$\begin{cases} v_\tau = \frac{1}{2} \left(v_{\hat{x}\hat{x}} + v_{\hat{r}\hat{r}} \right) \\[2mm] v(\hat{x}, \hat{r}, 0) = v_0(\hat{x}, \hat{r}) = e^{-AF_1(\hat{x}, \hat{r}) - BF_2(\hat{x}, \hat{r})} (e^{F_1(\hat{x}, \hat{r})} - K)^+, \end{cases} \tag{4.16}$$

where the function F_1 is such that $x = F_1(\hat{x}, \hat{r})$. Furthermore, let F_2 be such that $r = F_2(\hat{x}, \hat{r})$. The solution of (4.16) is given by

$$v(\hat{x}, \hat{r}, \tau) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{\frac{(\hat{x} - x_1)^2 + (\hat{r} - r_1)^2}{-2\tau}} v_0(\hat{x}, \hat{r}) \ dx_1 \ dr_1.$$

This allows us to compute

$$V(x, r, t, \epsilon = 0) = e^{Ax + Br + Ct} v \left(F_1^{-1}(x, r), F_2^{-1}(x, r), T - t \right).$$

The 0th and the 2nd term of the asymptotic expansion are given by

$$V_0 = V(x, r, t, \epsilon = 0)|_{\delta = 0} \tag{4.17}$$

and

$$V_2 = \lim_{\delta \to 0} \frac{V(\epsilon = 0) - V_0}{\sqrt{\delta}}. \tag{4.18}$$

We do not derive more explicit formulae for V_0 and V_2. We only mention that V_0 satisfies the normal Black-Scholes equation with volatility $\sigma = \bar{\sigma}$ and interest rate equal to the initial interest rate $r(t = 0) = r_0$.

4.4.4 The limit $\delta \to 0$

This section deals with the case $0 < \delta \ll \epsilon \ll 1$. We first let δ tend to 0 in (4.9) and then analyse the resulting PDE for small ϵ via singular perturbation techniques. As δ tends to 0, (4.9) reduces to

$$V_t + \frac{\sigma^2}{2} V_{xx} + \frac{\bar{\eta}^2}{2\epsilon} V_{\sigma\sigma} + \sigma \frac{\bar{\eta}}{\sqrt{\epsilon}} \rho_{S\sigma} V_{x\sigma} + \frac{\bar{\kappa}}{\epsilon} (\bar{\sigma} - \sigma) V_\sigma + \left(r_0 - \frac{\sigma^2}{2} \right) V_x - r_0 V = 0, \tag{4.19}$$

where $r_0 = r(t = 0)$ is the initial condition of the interest rate. As mentioned before, $\delta = 0$ means that the interest rate is constant at leading order on short timescales. Therefore, we take r equal to its initial value r_0.

We can now use known results that can be found, for instance, in [5], Section 5 of [11] and Section 4.4.2 of [12]. The authors apply singular perturbation techniques to a PDE

nearly identical to (4.19). It is worth mentioning that the analysis in Section 5 of [11] is very clear and a brief summary of the general perturbation procedure can be found in Section 2.6.2 of [18]. For simplicity, we assume that there is no market price of volatility risk. Hence, we conclude that

$$V_1 = -(T - t) \left(\frac{\bar{\eta} \rho_{S\sigma}}{2} \langle \sigma \partial_\sigma \phi \rangle S \partial_S \left(S^2 \partial_S^2 \right) \right) V_0, \tag{4.20}$$

where ϕ solves

$$\left(\frac{\bar{\eta}^2}{2} \partial_\sigma^2 + (\bar{\sigma} - \sigma) \partial_\sigma \right) \phi = \sigma^2 - \bar{\sigma}^2$$

and is chosen in such a way that V_1 satisfies the boundary conditions. Notice that $< . >$ is defined by

$$< f >= \int_{-\infty}^{\infty} f \frac{1}{\sqrt{\pi} \bar{\eta}} e^{-(\bar{\sigma} - \sigma)^2 / \bar{\eta}^2} \mathrm{d}\sigma.$$

In (4.20), V_0 is the solution to the normal Black-Scholes equation with average volatility $\bar{\sigma}$ and interest rate $r = r_0$. This results from arguments similar to those mentioned in the previous section.

4.5 Main result. Discussion

The main result of our paper is the expansion given by

$$V = V_0 + \sqrt{\epsilon} V_1 + \sqrt{\delta} V_2 + \mathcal{O}\left(\delta, \epsilon\right), \tag{4.21}$$

where V_0 solves the normal Black-Scholes equation with average volatility $\bar{\sigma}$ and the interest rate $r = r(t = 0) = r_0$, V_2 is given by (4.18) and V_1 is given by (4.20).

We have set a first step in applying existing perturbation methods to equation (4.2). Clearly more work has to be done especially concerning the calibration of the approximate solution (4.21) to real market data. If the approximation turns out to be not accurate enough, then one can look at some of the higher order terms (hoping to come closer to what happens in reality). We expect that the analysis of [5] can be extended in this direction. It is expected that evaluation of the approximate solution is much faster than solving the PDE, however there is a tradeoff between speed and accuracy. Once calibration with market data has been performed more can be said about improvements in the speed of computation.

In [5], the authors interpret the corrections to the leading order Black-Scholes approximation in terms of the Greeks (sensitivities). We expect that an intuitive interpretation of the correction factors can give further insight.

Using two small parameters instead of a single one offers *flexibility.* Instead of having two small parameters δ and ϵ one may be tempted to deal with a single one, i.e. $\delta = \mathcal{O}(\epsilon)$. However, we expect this later choice to essentially complicate the perturbation analysis.

We want to stress the fact that the validity of the formal perturbation approach is restricted by the conditions under which the pricing PDE with the imposed initial and boundary conditions is well-posed. It would be particularly interesting to study the effect of the degeneracy in the coefficients of the 2nd order derivatives on the solution of the PDE. Another open question is: What happens with the well-posedness of the model, and hence, with the approximate solution (4.21) if other boundary conditions are chosen instead of (4.5).

A completely different modeling approach is the so called random field approach. Let us sketch a very simple version of this idea. Consider the SDE $dS_t = rS_t dt + \sigma S_t dW_t^S$ and, for the moment, let σ and r be given constants. The Fokker-Planck equation for the probability distribution p of variables S and t is given by

$$\frac{\partial p}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 Sp}{\partial S^2} - \mu\frac{\partial Sp}{\partial S}.$$

If we now take μ and σ random in the above Fokker-Planck equation, then we are immediately led to *random fields.* Perturbation methods can also be applied to the resulting PDE; see, for instance, [6, 7, 15] and references therein.

We have been surprised that the seemingly straightforward problem that we addressed happened to be a box of Pandora, leaving open a lot of relevant mathematical problems of which this project is not the right framework to elaborate on. Particularly, we would like to stress that the proposed methods have not been tested at all and large deviations from reality may have been neglected.

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Bibliography

[1] Y. Achdou, B. Franchi, and N. Tchou. A partial differential equation connected to option pricing with stochastic volatility: regularity results and discretization. *Mathematics of Computation,* 74(251):1291–1322, 2004.

[2] Y. Achdou and N. Tchou. Variational analysis for the stochastic Black and Scholes equation with stochastic volatility. *Mathematical Modelling and Numerical Analysis*, 36(3):373–395, 2002.

[3] M. Böhm and R. E. Showalter. Diffusion in fissured media. *SIAM J. Math. Anal.*, 16(3):500–509, 1985.

[4] L. C. Evans. An Introduction to Stochastic Differential Equations. Department of Mathematics, UC Berkley, 2008.

[5] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna. Multiscale stochastic volatility asymptotics. *Siam J. Multiscale Model. Simul.*, 2(1):22–42 (electronic), 2003.

[6] C. W. Gardiner. *Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences*, volume 13 of *Springer Series in Synergetics*. Springer-Verlag, Berlin, third edition, 2004.

[7] J. Grasman and O. A. van Herwaarden. *Asymptotic Methods for the Fokker-Planck Equation and the Exit Problem in Applications.* Springer Series in Synergetics. Springer-Verlag, Berlin, 1999.

[8] L. A. Grzelak and C. W. Oosterlee. On the Heston model with stochastic interest rates. *SSRN: http://ssrn.com/abstract=1382902*, 2009.

[9] L.A. Grzelak, Oosterlee C.W., and S. van Weeren. Extension of Stochastic Volatility Equity Models with Hull-White Interest Rate Process. *Technical Report 2008 TUDelft. Available at: http://ssrn.com/abstract=1344959.*

[10] A. Haastrecht, R. Lord, A. Pelsser, and D. Schrager. Pricing Long-Maturity Equity and FX Derivatives with Stochastic Interest and Stochastic Volatility. *SSRN: http://ssrn.com/abstract=1125590*, 2008.

[11] S. Howison. Matched asymptotic expansions in financial engineering. *J. Engrg. Math.*, 53(3-4):385–406, 2005.

[12] S. Howison, A. Rafailidis, and H. Rasmussen. On the pricing and hedging of volatility derivatives. *Applied Mathematical Finance*, 11(4):317–346, 2004.

[13] P.-L. Lions and M. Musiela. Correlations and bounds for stochastic volatility models. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 24(1):1–16, 2007.

[14] B. Øksendal. *Stochastic Differential Equations*. Universitext. Springer-Verlag, Berlin, sixth edition, 2003. An Introduction with Applications.

[15] D. Repplinger. *Pricing of Bond Options*, volume 615 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 2008. Unspanned Stochastic Volatility and Random Field Models.

[16] L. O. Scott. Option pricing when the variance changes randomly: theory, estimation, and an application. *The Journal of Financial and Quantitative Analysis*, 22(4):419–438, 1987.

[17] P. Wilmott, S. Howison, and J. Dewynne. *The Mathematics of Financial Derivatives*. Cambridge University Press, Cambridge, 1995. A Student Introduction.

[18] X. Zhou. *Application of Perturbation Methods to Modeling Correlated Defaults in Financial Markets*. PhD thesis, North Carolina State University, 2006.