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Fouling Layers on Heat Exchangers with Temperature Dependent Viscosity.

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1. Introduction and Preview.

This report describes a mathematical model for the fluid motion and the heat transfer in a heat exchanger which gradually becomes clogged by particle deposition on the walls.

The heat exchanger consists of 170 tubes arranged in a hexagonal grid, with saturated steam passing along their insides and condensing so that its latent heat is transferred to the fuel oil which thus rises significantly in temperature. Deposits are formed on the outside of the tubes and they form a fouling layer which increases the resistance to heat transfer.

The problem consists of first finding a model which can be used to evaluate the amount of fouling in the heat exchanger assuming that we know the inlet and outlet temperatures of the oil, its mass flow rate, and the steam temperature. Secondly we wish to develop a model to predict the formation of the fouling layer.

As a first step we write down conservation equations for the fluid motion and the heat transfer. As a simplifying approximation we model the tube bundle as a parallel-plate heat exchanger. For the present we ignore the fact that the flow paths through the system are probably complicated due the presence of baffles etc. Evaluation of the Reynolds and Peclet numbers allows us to make use of 'lubrication theory' approximations. The chemical mechanism responsible for fouling is uncertain, and accordingly we propose several models for this, deferring a selection until appropriate data is available. We also mention briefly the numerical techniques we feel would be best suited to efficient solution of the proposed models.

To reduce the system of parabolic equations to a more tractable format we derive some averaged equations. These may prove useful in developing a simple model that could be used to calibrate on-line instrumentation for monitoring the heat exchanger fouling process as a function of time.

Lastly, we discuss the conclusions drawn from the work so far and highlight the major requirements for additional data. A summary of feasible future extensions is provided.

An appendix is provided to explain in detail the geometrical considerations involved in our choice of a parallel-plate analogue. In another appendix we discuss the present model in light of a similar problem considered previously by J.R. Ockendon and H. Ockendon.

2. Governing Equations for fluid regime.

For reasons of tractability we consider a 2 dimensional problem by considering a parallel plate heat exchanger. This approximation simplifies the mathematics and computational requirements greatly while retaining the essential elements of the problem. The dimensions of the parallel plates are chosen to preserve the mass flux and velocity of the oil, and are considered in more detail in appendix 1.

The fouling process requires about two years to transform a new heat exchanger into a terminally clogged state, so that on the time scale of a fluid element passing through the system we may treat the geometry as static. Accordingly we consider the steady-state incompressible Navier-Stokes equations with temperature dependent viscosity

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2.1}$$

$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right), \tag{2.2}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \tag{2.3}$$

The heat equation, neglecting viscous dissipation, is

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right), \tag{2.4}$$

where T is the oil temperature, and we have written ρ for the oil density, c_p as the specific heat, k as the thermal conductivity, and $\mu(T)$ as the temperature-dependent oil viscosity. We use the usual lubrication approximations to scale the variables as follows

$$x\sim L,\qquad y\sim h,\qquad u\sim u_o,\qquad v\sim v_o,\qquad T\sim T_o,\qquad \rho\sim\rho_o,\qquad \mu\sim\mu_o, \tag{2.5}$$

where L is the length of the heat exchanger, u_o is the mean fluid velocity, h is the distance between the surface of the fouling and the midpoint of the parallel plates, $v_o=\varepsilon u_o$, $P_o=\mu_o u_o/\varepsilon h$ and μ_o is the viscosity at some typical temperature T_o. This gives

$$\varepsilon\mathrm{Re}\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial P}{\partial x}+\varepsilon^2\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{2.6}$$

$$\varepsilon^2\mathrm{Re}\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial P}{\partial y}+\varepsilon^4\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\varepsilon^2\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right), \tag{2.7}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0,\qquad \varepsilon\mathrm{Pe}\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\varepsilon^2\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}, \tag{2.8, 9}$$

where

$$\mathrm{Re}=\frac{\rho u_o h}{\mu_o},\qquad \mathrm{Pe}=\frac{\rho c_p u_o h}{k},\qquad \varepsilon=\frac{h}{L}. \tag{2.10}$$

We can now estimate these important parameters. First, taking $h=0.25\,\mathrm{cm}$, and $L=350\,\mathrm{cm}$ gives

$$\varepsilon=7.14\times10^{-4}\sim10^{-3}. \tag{2.11}$$

For Re we use $\rho = 0.75\,\mathrm{g\,cm^{-3}}$, $u_o = 79\,\mathrm{cm\,s^{-1}}$, $\nu_o = 0.3$ Stokes (based on inlet temperature of $70\,^oC$) to give

$$\mathrm{Re}_{in} \sim 66, \qquad \mathrm{Re}_{out} \sim 329, \tag{2.12,13}$$

so that Re varies by a factor of 5 throughout the system, with the overall bounds

$$3 \times 10^{-5} < \varepsilon^2 \mathrm{Re} < 2 \times 10^{-4}, \tag{2.14}$$

allowing us to neglect ε^2 and ε^2Re terms. To estimate the Peclet number we use $\rho = 0.75\,\mathrm{g\,cm^{-3}}$, $k_o = 0.12\,\mathrm{W\,m^{-1}\,^oC^{-1}}$, $c_p = 1800\,\mathrm{J\,Kg^{-1}\,^oC^{-1}}$, yielding

$$\mathrm{Pe} = \frac{\rho c_p u_o h}{k_o} \sim 2.2 \times 10^4, \qquad \varepsilon\mathrm{Pe} \sim 16. \tag{2.15}$$

The simplified model, with $\varepsilon\mathrm{Pe} \sim 16$, is

$$\frac{\partial P}{\partial x} = \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right), \qquad \frac{\partial P}{\partial y} = 0, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.16,17,18}$$

$$\varepsilon\mathrm{Pe}\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial^2 T}{\partial y^2}. \tag{2.19}$$

P can be eliminated from these governing equations, so we must now look only for boundary conditions for u, v and T. On the channel mid-plane $y = 0$ we have a line of symmetry and hence

$$y = 0 \quad \Rightarrow \quad \frac{\partial u}{\partial y} = 0, \qquad v = 0, \qquad \frac{\partial T}{\partial y} = 0. \tag{2.20,21,22}$$

On the layer of deposit, with surface at $y = h(x)$, we have

$$y = h \quad \Rightarrow \quad u = 0, \qquad v = 0, \qquad -k\frac{\partial T}{\partial y} = H(T - T_s), \tag{2.23,24,25}$$

assuming no-slip conditions there. The heat-flux condition incorporates several assumptions which are discussed below.

Consider a section through the tube wall, and attempt to find a single composite conductivity due to both the metal of the tubes and any fouling adhering to them. Suppose the tube wall is of thickness d with thermal conductivity k_T, and that the fouling layer is thickness $l(x)$ with conductivity k_F. The heat flux Q can be calculated by assuming 1-dimensional quasi-steady flow, to give the relations

$$Q = \frac{k_T}{d}(T_s - T_i) = \frac{k_F}{l(x)}(T_i - T) = H(T_s - T) = k\frac{\partial T}{\partial y}, \tag{2.26}$$

where all the variables are in dimensional form, T is the temperature at the oil-fouling interface, T_i is the temperature at the fouling-wall interface, and T_s is the known steam temperature, and H is the net heat transfer coefficient across the tube and fouling material. Eliminating T_i gives the relation

$$\frac{\partial T}{\partial y} = \lambda(T_s - T), \qquad \frac{1}{\lambda} = k\left(\frac{d}{k_T} + \frac{l(x)}{k_F}\right). \tag{2.27,28}$$

Note that when the fouling layer is absent λ represents the metal's resistance to heat transfer, but as the fouling layer thickens λ becomes dominated by the deposited material.

3. Governing equations for layer formation.

The exact mechanism responsible for the fouling effect is uncertain and we present here three models, each appropriate to different chemical processes. The information available at the Study Group was insufficient to enable us to select the most appropriate choice; and if more detail is impossible then all three could be solved and the results compared with experimental data.

The fouling process takes place on a much slower time-scale than the fluid processes previously discussed and may be treated as pseudo-steady state on a time-scale of days. Numerically this allows us to de-couple the two regimes to a large extent.

Mechanism 1. It is suggested that small molecular complexes (fouling pre-cursors) are formed within the oil when heated, and that these particles adhere to the walls whenever the fluid motion brings them sufficiently near. Taking C as the concentration of pre-cursors in the bulk oil, this could be modelled by the equation

$$\frac{\partial}{\partial x}(uC) + \frac{\partial}{\partial y}(vC) = \frac{\partial}{\partial y}\left(D\frac{\partial C}{\partial y}\right) + \Omega(T), \tag{3.1}$$

with boundary conditions

$$C\big]_{x=0} = 0, \qquad \left.\frac{\partial C}{\partial y}\right|_{y=0} = 0, \qquad -vC + D\frac{\partial C}{\partial y} = \zeta C\big]_{y=h}, \tag{3.2, 3, 4}$$

where $\Omega(T)$ is the temperature dependent body formation rate, h is the present thickness of the deposit, and ζ is a coefficient representing the rate at which pre-cursors adhere to the wall. The diffusion coefficient D should also include an effective dispersive component to allow for swirling within the oil. In this scenario the rate of wall growth is given by

$$\frac{dh}{dt} = -\gamma\zeta C\big]_{y=h}, \tag{3.5}$$

where γ is a scale factor dependent on the molecular size of the fouling pre-cursors, and we note that h decreases as the channel becomes increasingly clogged.

Mechanism 2. It is possible that the wall growth occurs primarily due to some polymerisation reaction occurring on the wall surface between the existing fouling and the oil, and that the pre-cursor formation is irrelevant. In this case we do not require an equation for C, and instead have the single temperature-dependent growth equation

$$\frac{dh}{dt} = -f(T), \tag{3.6}$$

where, lacking any better evidence, we would assume f to have a modified Arrhenius form, for instance $f \propto T^{\alpha}\exp(-E/RT)$.

Mechanism 3. The real growth mechanism is almost certainly a combination of the above two suggestions. Mechanism 1 assumes that adherence is purely passive, whereas mechanism 2 assumes that there are always sufficient pre-cursors-available for 'burn-on' at the wall. If neither mechanism can be shown *a priori* to be dominant, we may combine them by taking $\zeta = \zeta(T)$ in equation (3.4) so that the proportion of pre-cursors adhering to the walls is temperature dependent.

4. Numerical Formulation.

We may eliminate the pressure P from the fluid velocity equations by direct differentiation and substitution to get

$$\frac{\partial^2}{\partial y^2}\left(\mu\frac{\partial u}{\partial y}\right) = 0, \tag{4.1}$$

and upon introducing the usual Cartesian stream function ψ such that $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$, our equations may be written in the form

$$\frac{\partial^2}{\partial y^2}\left(\mu(T)\frac{\partial^2 \psi}{\partial y^2}\right) = 0, \qquad \epsilon \mathrm{Pe}\left(\frac{\partial \psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial T}{\partial y}\right) = \frac{\partial^2 T}{\partial y^2}, \tag{4.2,3}$$

subject to the boundary conditions

$$y = 0 \quad \Rightarrow \quad \psi = 0, \qquad \frac{\partial^2 \psi}{\partial y^2} = 0, \qquad \frac{\partial T}{\partial y} = 0 \tag{4.4,5,6}$$

$$y = h(x) \quad \Rightarrow \quad \psi = 1, \qquad \frac{\partial T}{\partial y} = \lambda(T_s - T), \tag{4.7,8,9}$$

where the latter condition on ψ at $y = 0$ is equivalent to $\frac{\partial \psi}{\partial x} = 0 = \frac{\partial \psi}{\partial y}$. Since the heat equation is parabolic we must specify 'initial' data at $y = 0$ for T and ψ, and suggest

$$x = 0 \quad \Rightarrow \quad \psi = \frac{3}{2}\left(y - \frac{1}{3}y^3\right), \qquad T = T_0, \tag{4.10,11}$$

which corresponds to a parabolic velocity profile entering at a uniform temperature. The fouling layer will probably grow non-uniformly along the length of the plate, and this can be accommodated without difficulty by the change of variable

$$z = \frac{y}{h(x)}, \tag{4.12}$$

mapping the region onto a rectangle and slightly modifying the coefficients of the governing equations.

The numerical scheme suggested for the fluid velocity would be of the marching type, discretising x with nodes $i = 0, \ldots, N_x$ and z with nodes $j = 0, \ldots, N_y$. Taking $i = 0$ as the entry data one would use forward differences to evaluate $\frac{\partial T}{\partial x}$ and $\frac{\partial \psi}{\partial x}$ at $i = n$ in terms of the values at $i = n-1, n$. This would produce a non-linear set of difference equations in the z-direction for the T and ψ profiles at x which could be solved by an iterative Newton-Raphson method. After convergence at each value of i the scheme advances to $i+1$. Such an implicit scheme would be stable, with no limit on the size of the x-discretisation as far as stability is concerned. It may be necessary to incorporate adaptive gridding due to the uncertainty of the location of the thermal boundary layers.

The equations for T and C (if required) are parabolic, and an excellent method would be a pair of finite element routines using triangular basis elements. The nature of these equations, being linear except perhaps in the source terms, means that a fully implicit solution could be implemented very efficiently.

Based on these suggestions, a good program would be structured as three primary subroutines linked through an overall control handler using error estimation techniques. The subroutines would each solve one of the equations for fluid velocity, fluid temperature or fouling deposition. The first two routines must be coupled iteratively because of the temperature dependence of the fluid viscosity, whereas the third component may be evaluated less frequently — say at 'real time' intervals of about one week (since the whole fouling process goes from start to completion in approximately two years).

5. Averaged Equations.

It is possible to make some estimates of the behaviours likely to be found in our scenario by averaging the governing equations across the gap width. We introduce a new variable z such that

$$z = \frac{y}{h(x)}, \tag{5.1}$$

where $g(x) + h(x) = 1$, which maps the problem onto the region $[0,1] \times [0,L]$. The continuity equation becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \Rightarrow \qquad \frac{\partial u}{\partial x} - \frac{h'(x)z}{h(x)}\frac{\partial u}{\partial z} + \frac{1}{h(x)}\frac{\partial v}{\partial z} = 0, \tag{5.2}$$

which may be integrated w.r.t. z across $[0,1]$ to get

$$\int_0^1 \frac{\partial u}{\partial x}\,dz - \frac{h'(x)}{h(x)}\int_0^1 z\frac{\partial u}{\partial z}\,dz + \frac{1}{h(x)}\int_0^1 \frac{\partial v}{\partial z}\,dz = 0. \tag{5.3}$$

Writing $\overline{u} = \int_0^1 u\,dz$, and using the result $\frac{\partial}{\partial z}(uz) - u = z\frac{\partial u}{\partial z}$, together with the boundary conditions $v = 0$ at $z = 0, 1$ we see that

$$\frac{\partial \overline{u}}{\partial x} + \frac{h'}{h}\overline{u} = \frac{1}{h}\frac{\partial}{\partial x}[h\overline{u}] = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x}[h\overline{u}] = 0. \tag{5.4}$$

The momentum equation $P_x = (\mu u_y)_y$ becomes

$$h^2\frac{\partial P}{\partial x} = \frac{\partial}{\partial z}\left(\mu\frac{\partial u}{\partial z}\right), \tag{5.5}$$

where $\frac{\partial P}{\partial z} = 0$. Using $\frac{\partial u}{\partial z} = 0$ at $z = 0$ we integrate this from 0 to z to give

$$h^2\frac{\partial P}{\partial z} = \mu\frac{\partial u}{\partial z} - \mu(T_0)\frac{\partial u}{\partial z}\Big|_0 = \mu\frac{\partial u}{\partial z}, \tag{5.6}$$

and averaging over $z \in [0,1]$ produces the equation

$$\begin{aligned} h^2\frac{\partial P}{\partial x}\int_0^1 z\,dz &= \int_0^1 \mu\frac{\partial u}{\partial z}\,dz \\ &= [\mu u]_0^1 - \int_0^1 u\frac{\partial \mu}{\partial z}\,dz = -\mu_0 u_0 - \overline{u\frac{\partial \mu}{\partial z}} \end{aligned} \tag{5.7}$$

which gives us the average momentum equation

$$\overline{u\frac{\partial \mu}{\partial z}} + \mu_o u_o + \frac{1}{2}h^2\frac{\partial P}{\partial x} = 0; \tag{5.8}$$

here u_c, μ_c and T_c are the centreline velocity, viscosity and temperature, and $\frac{\partial \mu}{\partial z} = \frac{\partial \mu}{\partial T}\frac{\partial T}{\partial z}$ is the change of viscosity w.r.t. z.

Finally, consider the heat equation. Noting that $T(u_x + v_y) = 0$, we may write it in conservative form as

$$\varepsilon\mathrm{Pe}\left(\frac{\partial}{\partial x}(uT) + \frac{\partial}{\partial y}(vT)\right) = \frac{\partial^2 T}{\partial y^2}, \tag{5.9}$$

$$\varepsilon\mathrm{Pe}\left(\frac{\partial(uT)}{\partial x} - \frac{h'(x)z}{h(x)}\frac{\partial(vT)}{\partial z} + \frac{1}{h(x)}\frac{\partial(vT)}{\partial z}\right) = \frac{1}{h(x)^2}\frac{\partial^2 T}{\partial z^2}. \tag{5.10}$$

Integrating this over $z \in [0,1]$ using the relation $z\partial(uT)/\partial z = \partial(zuT)/\partial z - uT$, yields the averaged heat equation

$$\varepsilon \mathrm{Pe}\left(\frac{\partial}{\partial x}\overline{(uT)} + \overline{(uT)}\frac{h'(x)}{h(x)}\right) = \frac{1}{h(x)^2}\frac{\partial T}{\partial z}\Big|_{z=1} = \frac{\lambda}{h(x)^2}(T_s - T), \qquad ()$$

where $\lambda = K/(1 - h(x) + l_0)$ as usual.

These results may be exploited by assuming that u and T have some particular parametric form, performing the averages, then solving to find particular values for the parameters. The use of this type of model is that its simple 1-dimensional nature makes it amenable to very rapid numerical calculations, so that it could be used as the basis for developing real-time instrumentation to monitor fouling — provided the more sophisticated model is first used to relate the thickness of the layer to observable properties such as the outlet pressure and temperature.

Conclusion.

In this report we have developed a model for the coupled flow of oil and heat within the heat exchanger, and a series of alternate models to describe the fouling process responsible for gradual degradation of the system's performance. A averaged model was derived from these fuller descriptions.

In order to develop the fouling model further, additional experimental data is required. We were unable to examine a heat exchanger to see which regions became clogged fastest and thus partially validate our model. The exact chemical mechanism should be high-lighted and its kinetics detailed.

Given the above data, we would suggest that the computer programs described in section 4 should be implemented and calibrated against the known data. This would allow a variety of simulations to be run to explore the effects of varying plate spacings and lengths — corresponding to different heat exchanger geometries. It may be possible to estimate the degree of fouling of a heat exchanger by measuring only the oil inlet pressure and the steam temperature which would allow automatic equipment to be designed to indicate when performance became unacceptably degraded. The full numerical solutions could perhaps be employed in the averaged equations to produce simple 'rule-of-thumb' tables covering this situation. We estimate that to go from beginning the programming work to graphical output and comparison of different operating strategies represents about four months full-time work.

Appendix 1 : Geometry.

The total oil mass flow rate Q_{oil} is given, and we want some estimate of the oil velocity u in the tube bundle. We define a local velocity by

$$u = \frac{Q_{oil}}{\rho_{oil} A_{tot}}, \qquad (a1.1)$$

where A_{tot} is the total area between the tubes available for fluid flow, and ρ_{oil} is the oil's density.

Take a typical unit cell as consisting of the region delineated by lines drawn from the centres of any three adjacent tubes, and denote its area by A_c. We assume that all of the tubes together form a hexagonal array. We need to calculate the total number of tubes and unit cells for a given value of N_o (the number of tubes along the mid-plane of the pattern).

Let N_{tubes} be the total number of tubes in a hexagonal pattern. From simple geometry we get

$$N_{tubes} = N_o + 2\{(N_o-1)+(N_0-2)+\ldots+(N_s+1)+N_s\} = N_o+\{N_o(N_o-1)-N_s(N_s-1)\} = \frac{3N_o^2+1}{4}, \qquad (a1.2)$$

where we have used the notation $N_s = (N_o - 1)/2 + 1$. The total number of unit cells is N_c, which is given by

$$N_c = 2(N_o-1)+2(N_s-1)+4\{(N_o-2)+(N_o-3)+\ldots+(N_s+1)+(N_s)\} = 2(N_o-1)^2-2(N_s-1)^2 = \frac{3}{2}(N_o-1)^2, \qquad (a1.3)$$

so that we have the relationships

$$N_o = \sqrt{\frac{4N_{tubes}-1}{3}}, \qquad N_c = \frac{3}{2}(N_o-1)^2. \qquad (a1.4,5)$$

We are give $N_{tubes} = 170$, so that $N_o \approx 15.04$, and for convenience we shall take $N_o = 15$, which corresponds to $N_{tubes} = 169$ and $N_c = 294$ cells, with each side of the pattern having $N_s = 8$ tubes. The total area is given by

$$A_{tot} = N_c A_c + 6(N_s - 1)A_p, \qquad (a1.6)$$

and consists of all the triangular unit cells plus a contribution from the peripheral areas adjacent to the outermost cells, denoted by A_p. Using $r = 0.635\,\mathrm{cm}$ and $h = 0.250\,\mathrm{cm}$ gives the following typical values

$$A_c = \frac{\sqrt{3}}{2}(0.635+0.250)^2-\frac{\pi}{2}(0.635)^2 = 0.045\,\mathrm{cm}^2, \qquad A_p = 2(0.635+0.250)(0.635)-\frac{\pi}{2}(0.635)^2 = 0.049\,\mathrm{cm}^2, \qquad (a1.7,8)$$

so that $A_{tot} = 33.807\,\mathrm{cm}^2$. Note that the peripheral area is apparently as significant as the inter-tube area. This emphasises that the details of the geometry are crucial if we are to obtain an accurate estimate of the total area and the local oil velocity.

We are given $Q_{oil} = 2000\,\mathrm{g\,s^{-1}}$, and ρ_{oil} is uncertain but should lie within the limits $0.59\mathrm{g\,cm^{-3}} \leq \rho_{oil} \leq 1\mathrm{g\,cm^{-3}}$. Choosing $\rho_{oil} = 0.75\mathrm{g\,cm^{-3}}$ for definiteness gives a value for u of

$$u = \frac{2000\,\mathrm{g\,s^{-1}}}{0.75\,\mathrm{g\,cm^{-3}} \times 33.8\,\mathrm{cm}^2} = 79\,\mathrm{cm\,s^{-1}}. \qquad (a1.9)$$

Based on the uncertainty in ρ_{oil}, u should lie within the range $60\,\mathrm{cm\,s^{-1}}$ to $120\,\mathrm{cm\,s^{-1}}$ — essentially a factor of 2.

Appendix 2 : A Previous Model.

The equations in Section 2 were examined in a different context by Ockendon & Ockendon (Jou.Fl.Mech., 1977, Vol.83, p177–190, "Variable viscosity flows in heated and cooled channels"), who made an analysis of entry flow into a channel with a step change in the wall temperature at the entry point.

Their model assumes an infinitely long channel such that the x-scaling is given by

$$L_0 = \frac{\rho c h^2 U}{k}, \qquad \overline{x} = x L_0, \tag{a2.1,2}$$

where $\overline{x}$ is the length, and x is its dimensionless equivalent. In our application the channel is of length L and we wish to think of x as

$$\frac{\overline{x}}{L} = \frac{x L_0}{L} = x \frac{\rho c h U}{k} \frac{h}{L} = x \mathrm{Pe} \varepsilon, \qquad x = x_{ock} \mathrm{Pe} \varepsilon. \tag{a2.3,4}$$

The important limit in the isothermal wall model is

$$x_{ock} = \theta \mu_o^2 \beta^2, \tag{a2.5}$$

where μ_o is the dimensionless viscosity at the wall and β is defined by

$$\mu(T) = \exp(-\beta T). \tag{a2.6}$$

Their model demonstrates the transition from an entry region where there is a sharp thermal gradient at the walls to a fully developed flow where this gradient has evened out throughout most of the fluid. According to this isothermal wall model, the fluid is heated so that it develops a slipping core region when

$$x_{ock} \sim \theta \mu_o^2 \beta^2 \approx 0.5, \tag{a2.7}$$

which in terms of our present scaling implies

$$x \approx 0.7, \tag{a2.8}$$

so that the slipping core region is about 70% of the tube length from the entry, and hence most of the flow regime is similar to the 'entry flow' with respect to the development of thermal boundary layers.

For our case it is also relevant to note that changing the wall boundary condition from isothermal to heat-flux could have some effect on the distinction between the entry and full developed flow regimes.