DEVELOPING AN UNDERSTANDING OF WASHING MACHINE DYNAMICS

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Abstract

This report summarises the findings of one of the problems presented at the 2006 MISG. The problem, presented by Fisher and Paykel, was to develop a better understanding of the dynamics of modern washing machines that use 'balance rings'. Balance rings are toroidal chambers that are partially filled with fluid. The washing machine is a complicated system so several simplified systems were examined instead of attempting to model the complete system. These approaches included

1 experimental work using a washing machine with transparent balance rings and a strobe-light to observe the fluid-flow in the balance rings,

2 developing a 2D model of a balance ring and examining the system behaviour,

3 developing a simplified 3D model of a washing machine, without balance rings or an out-of-balance mass, and examining the system behaviour,

4 developing a simplified model to describe waves that might occur in the fluid within the balance rings.

1. Introduction

1.1. The product

The industry partner manufactures washing machines which have a cylindrical drum (with a nominally vertical axis) suspended within a cuboid outer box (called the envelope). The drum has a non-rotating outer part and a rotating inner bowl where the clothes (load) are placed.

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The centre of mass of the load will not usually lie on the axis of symmetry of the inner drum and thus there will be an out of balance load (OOBL) when the drum rotates. This causes the drum to rotate about an axis other than its axis of symmetry and causes the motion to be eccentric.

The industry partner seeks to maximise the size of the inner drum (to maximise capacity) whilst minimising the size of the envelope (space required in laundry) and thus the clearance between the bowl and the envelope needs to be minimised (it is typically 25mm). This constrains the amount of eccentricity that can be tolerated before the drum collides with the envelope. Eccentricity is reduced by the use of balancing rings on the inner drum and by the design of the suspension system of the drum.

1.2. Objective

The industry partner seeks a more comprehensive understanding of the dynamics of the system and in particular what causes the motion to be eccentric. They have always been able to successfully design balance rings and suspension rods which keep the eccentricity to acceptable levels for all reasonable OOBLs but the design approach has been somewhat "trial and error".

1.3. The approach

A modern washing machine with balance rings is a complicated system so, rather than modelling the complete system, simplified models were developed for various parts of the system. Thus, several approaches were taken to gain a better understanding of the dynamics of a washing machine. The four main approaches were

- Experimental work was undertaken to observe the fluid flow in the balance rings of a washing machine at a range of speeds and with a range of out-of-balance loads. Fisher and Paykel supplied a washing machine with transparent balance rings to facilitate this experimentation.
- A simplified 2D[1]model of a washing machine, including balance rings and an out-of-balance load, was developed and examined.
- A simplified 3D[2] model of a washing machine, without balance rings or an out-of-balance load was developed.

[1]2D refers to the fact that the model is assumed to lie completely in the horizontal plane.
[2]3D refers to 3 spatial dimensions.

- A simplified model of the fluid in the balance rings was developed which included an estimate of the speed at which surface waves would travel in the balance rings.

These approaches are considered in the remainder of this report.

2. Experimental work

2.1. Introduction

Fisher and Paykel provided a washing machine with transparent balance rings and a strobe-light so that the group could observe the dynamics of the fluid within the rings.

2.2. Method and Main results

2.2.1 Background. Observations and the industry partner's experience suggest that eccentricity is relatively small during the low speeds of the wash cycle (say below 30rpm) and during the high speeds reached towards the end of the spin cycle (above 300rpm and up to approximately 1,000rpm) but that there are two speed ranges (traversed during the spin cycle) where the eccentricity can become large. The first occurs typically between 30 and 60rpm and across a narrow speed range (for any given configuration and load). At such low speeds the drive motor has abundant torque and the undesirable eccentric motion can be avoided by rapidly accelerating through the speed range where it would occur. The second speed range occurs typically between 150 and 300rpm and (for any given configuration and load) prevails over a wider speed range. This wide speed range combined with reduced motor drive torque available at these higher speeds mean that it is not possible to simply rapidly accelerate through the speed range and careful attention needs to be paid to the design of the suspension and the balance rings to achieve acceptable behaviour.

2.2.2 Observations at low speed. The low speed mode corresponds to the natural frequency of the swing mode of the drum hanging on the suspension rods (with or without drum rotation, the drum can simply swing from the suspension rods). This natural frequency was measured to be slightly below 1 Hz and the motion was excited at a rotational speed of approximately 50rpm at which the out of balance centrifugal force would excite the mode. The amplitude (and natural frequency) of the motion would change with suspension characteristics (stiffness and damping). However, the design of the suspension system is a compromise between minimising the amplitude of the motion of the drum relative to the envelope and minimising the transmission of high frequency vibrations (hence noise) from the drum to the envelope. Thus, though it would be possible to reduce the amplitude of the swing by stiffening the suspension or increasing the damping, both would tend to increase the transmission of high frequency vibrations to the envelope.

It should be noted that, at the speeds where the low-speed mode is observed, the effect of gravity is greater than the centripetal acceleration at the edge of the balance ring (due to the fact that the radius of the balance ring was 250mm). The simple model in the "Simple Theory" section of [1], though for a system with a massive base plate on the drum and with horizontal suspension connections to the envelope, illustrates this mode. A model to describe this behaviour for the modern designs (no massive base plate) could be developed relatively easily. Overall, the team suggest that the current suspension system design achieves a reasonable compromise between suppressing eccentric motion in this mode and minimising vibrations and noise at higher frequencies and that the solution of rapidly accelerating through the critical speed is a sensible one.

2.2.3 Observations at higher speed. Once higher speeds are reached, centrifugal forces dominate the system. This is because the radius of the balance ring is 250mm so that, at 135rpm, the centripetal acceleration is $5g$. Like a spinning top, the inner drum will tend to rotate about its principal axis of inertia (axis about which the moment of inertia is minimal). Due to the OOBL, the principal axis is dissimilar to the axis of symmetry and hence the motion is eccentric. The balance rings are able to partially counterbalance the OOBL and bring the principal axis closer to the axis of symmetry (thus reducing eccentricity) but it can be shown that they are not able to completely remove the eccentricity. The model in MISG 2000 [2] provides a good description of the operation of the balance rings and the resulting, reduced eccentricity. A key assumption of this 2-D model is that the axis of rotation will be vertical and stationary, thus the motion of the drum is a simple 1-degree of freedom (dof) rotation (albeit about an axis inclined to the axis of symmetry). Key characteristics of such a motion are that the gap (vertical and horizontal) between a fixed point on the envelope and the upper rim of the drum is sinusoidal through time with fixed amplitude and frequency equal to the rotational frequency of the inner drum and the length of the suspension rods is also sinusoidal through time with fixed amplitude and similar frequency. Further, for this motion, the water in the balance rings will be stationary with respect to the ring and its position and profile will not be speed dependant.

Observations of the motion suggest that these displacements are sinusoidal, that the balancing water is stationary with respect to the balance rings, and that the motion is the simple 1-dof motion predicted for both high speeds (say above 300rpm) and for speeds below a critical threshold (approximately 180rpm for the configuration of the experimental rig available during the workshop). However, at the speeds where the eccentricity is high (approximately 180rpm to 250rpm), the motion is not the simple 1-dof rotation described by the MISG 2000 model. This is evident as the gap between the drum and envelope does not vary sinusoidally and the water moves within the balance rings.

Thus, it appears that the assumption made in the MISG 2000 model about the axis of rotation being vertical and stationary is not valid at the speeds where the problem occurs and a richer model is required to describe the motion and understand the cause of the eccentricity. The two degrees of rotational freedom, suppressed in the MISG 2000 model, can be described as precession and nutation, in these degrees of freedom the axis of rotation of the spin will itself pitch and roll. When precession and nutation[1] occur, the eccentricity (of the drum's rotation viewed in a horizontal plane) is likely to increase. There is no fundamental reason why precession and nutation will not occur. A brief analysis of a simple model developed to describe the dynamics and permitting precession and nutation suggest that there are spin-speed dependant terms which significantly affect the precession and nutation behaviour at higher speeds. However, much further modelling and analysis is required to provide a complete explanation.

2.2.4 Effects of suspension, fluid motion and motor-drive.

The suspension system will play a vital role isolating and damping the resulting motion and it is possible that baffles which damp the motion of the water within the balance rings will also affect the motion. This is consistent with the industry partner's experience of significant design parameters which influence the eccentricity.

Further phenomena which were observed during the non-simple rotational motion were that there appeared to be standing waves between the baffles within the balance rings and that it appeared that the spin speed of the drum was not constant.

The spin speed is regulated by feedback control which manipulates the torque applied by the motor to drive the spin. Ideally, the spin speed will be tightly controlled so that variations in spin speed do not create

[1]Precession is when the axis of spin rotates so that it describes a cone shape. Nutation is when the angle that the cone of precession describes changes periodically.

undesirable motions, and potential disturbances to spin speed, caused by undesirable motions, are rejected by the control system.

It is not certain whether the standing waves in the balance ring sectors are an effect or a cause of the non-simple rotational motion. However, since the water is stationary within the balance rings when the rotational motion is simple (outside the 180 to 250rpm range), there would have to be some self-generating effect for the waves to be the cause of the non-simple motion as the spin speed entered the 180 to 250rpm range.

2.3. Summary

At high speed (say greater than 250rpm) and below a threshold (typically 180rpm), the motion is predominantly a simple 1dof rotation with the axis of rotation slightly tilted with respect to the axis of symmetry and slight (but acceptable) eccentricity. The balance rings operate well with the water stationary within them and the MISG 2000 model adequately describes the behaviour of the system. At intermediate speeds (say 180 to 250rpm) the motion is non-simple with excessive (potentially unacceptable) eccentricity. Three candidate causes have been identified:

1 The motion is naturally non-simple and precession and nutation naturally occur at these speeds.

2 Waves occur in the balance ring sectors and these force the rotational motion to be non-simple.

3 The spin speed control is ineffective and spin speed fluctuations are the cause.

The third potential cause is essentially a stand-alone issue and can be analysed and resolved independently. A richer model could be developed which describes the non-simple motion and how it is influenced by suspension system and balance ring design. This would be potentially useful to the industry partner as a design aid.

3. 2D model of a rigid-body system

3.1. Introduction

An analysis of a simplified 2D model of a balance ring was considered. This analysis illustrated the benefits of using balance rings to reduce vibration in a washing machine.

3.2. Main result

The analysis conducted by the group produced almost identical results to [2], so the reader is referred to that paper for a good summary of the 2D case.

4. 3D model of a rigid-body system

4.1. Introduction

A model of a 3D rigid-body washing machine was considered. Thus, the model was simplified by removing balance rings and out-of-balance loads. This approach was useful because it helped the group to determine if certain effects were likely to be due to rigid-body motion of the washing machine or due to dynamic effects in the balance rings. Although a simplified model was considered, the equations of motion were found to be non-linear and highly coupled.

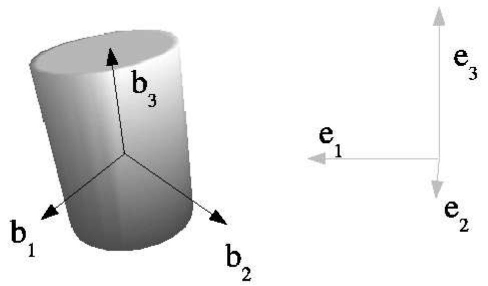

Figure 1. Sets of unit vectors used in the model

4.2. Preliminaries

First, some notation must be defined. Figure 1 shows a set of unit vectors $\mathcal{E} = \{e_1, e_2, e_3\}$ fixed in the earth and a set of unit vectors $\mathcal{B} = \{b_1, b_2, b_3\}$ such that b_3 is in the direction of the axis of symmetry of the bowl and b_2 is perpendicular to e_3. The steps in the transformation required to get from $\mathcal{E}$ to $\mathcal{B}$ are:

1 Start with the axis set defined by $\mathcal{E}$ and rotate the axes by an angle ϕ about the e_3 axis. Call this intermediate axis set $\mathcal{C} = \{c_1, c_2, c_3\}$.

2 Then rotate the axes by an angle θ about c_2. This axis set is $\mathcal{B}$.

Thus, the relationship between $\mathcal{B}$ and $\mathcal{E}$ is given by (1).

$$b = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi\cos\theta & \cos\phi\cos\theta & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix} e. \qquad (1)$$

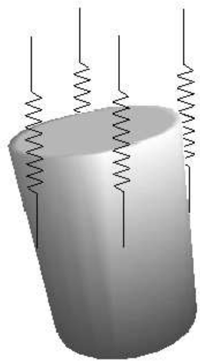

Figure 2. Sketch of the bowl being supported by springs

Now, suppose the bowl is spinning with an angular speed of $\dot{\psi}$ about the b_3 axis. Therefore, it can be seen that $\dot{\phi}$ is the precession speed and $\dot{\theta}$ is the nutation speed of the motion.
Note that, in Figure 1, the vector b_2 is in the plane defined by e_1 and e_2.
Figure 2 shows that the bowl is supported by four springs which counteract the weight of the bowl. There are seven assumptions that are used for the model. They are:

1 The springs that support the washing machine bowl are oriented vertically and are assumed to be long so that they remain vertical at all times. See Figure 2.

2 There is no damping or friction in the system.

3 The bowl is allowed to spin freely and there are no external moments such as motor torque acting on the bowl.

4 The vertical bounce mode of the bowl and any horizontal translations of the bowl are ignored.

5 The height of the centre of gravity of the washing machine bowl is the same as the height where the springs attach to the bowl when the bowl is at rest in its equilibrium position.

6 There is no out-of balance mass so that the moment of inertia about the b_3 axis is I_3 and the moments of inertia about the b_1 and b_2 axes are both $I_\perp$. Furthermore, $I_3 < I_\perp$. See Figure 1.

7 The washing machine bowl is a rigid body so balance-rings and fluid in the bowl are not included in the model.

Although the assumptions make the model considerably simpler than a real washing machine, the simplified model does contain enough detail to provide a good amount of insight into the likely behaviour of a real machine. In particular, precession and nutation are permitted in this model.

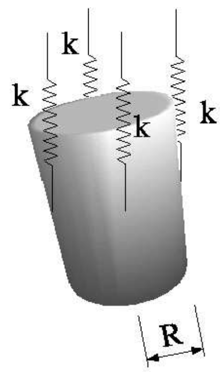

Figure 3. Pictorial representation of the radius of the bowl and stiffness of the springs

4.3. Main Result

This section presents the equations of motion for the washing machine (cf. Figure 3).

Theorem 1 *The equations of motion are given by the set of non-linear differential equations (2), (3), (4).*

$$\ddot{\theta} = \frac{\sin\theta}{I_\perp}\left((I_\perp - I_3)\cos\theta\dot{\phi}^2 - I_3\dot{\psi}\dot{\phi} - 2Rk\cos\theta\right), \tag{2}$$

$$\ddot{\phi} = -\frac{(2I_\perp + I_3)\cos\theta}{I_\perp \sin\theta}\dot{\phi}\dot{\theta}, \tag{3}$$

$$\ddot{\psi} = \left(\frac{(2I_\perp + I_3)\cos\theta}{I_\perp \sin\theta} + \sin\theta\right)\dot{\phi}\dot{\theta}. \tag{4}$$

Proof: The potential energy U stored in the springs is given by

$$U = Rk\sin^2\theta \tag{5}$$

The kinetic energy T is given by summing the kinetic energy of rotation in the direction of each of the principal moments of inertia. These principal moments of inertia are in the directions of the three unit vectors b_1, b_2 and b_3.

To calculate the kinetic energy, first consider the angular velocity ω where

$$\omega = \dot{\theta}b_1 + \dot{\phi}e_3 + \dot{\psi}b_3 \tag{6}$$

Rearranging (1) and substituting for e_3 in (6) gives

$$\omega = \dot{\theta}b_1 + \dot{\phi}\sin\theta b_2 + \left(\dot{\psi} + \dot{\phi}\cos\theta\right)b_3. \tag{7}$$

Thus, the kinetic energy is

$$T = \frac{1}{2}I_\perp\dot{\theta}^2 + \frac{1}{2}I_\perp\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2. \tag{8}$$

Equations (5) and (8) may be used to calculate the Lagrangian,

$$\begin{aligned} L &= T - U && (9)\\ &= \frac{1}{2}I_\perp\dot{\theta}^2 + \frac{1}{2}I_\perp\dot{\phi}^2\sin^2\theta + \frac{1}{2}I_3\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - Rk\sin^2\theta. && (10)\end{aligned}$$

Hamilton's principle of least action gives the Euler-Lagrange equations for the system as (11) to (13).

$$\begin{aligned} 0 &= \frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}}, && (11)\\ 0 &= \frac{\partial L}{\partial\phi} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}}, && (12)\\ 0 &= \frac{\partial L}{\partial\psi} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\psi}}. && (13)\end{aligned}$$

The following partial derivatives are useful in the sequel:

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} &= \sin\theta\left(\left(I_{\perp}-I_3\right)\dot{\phi}^2\cos\theta-\dot{\psi}\dot{\phi}I_3-2Rk\cos\theta\right), && (14)\\
\frac{\partial L}{\partial \phi} &= 0, && (15)\\
\frac{\partial L}{\partial \psi} &= 0, && (16)\\
\frac{\partial L}{\partial \dot{\theta}} &= I_{\perp}\dot{\theta}, && (17)\\
\frac{\partial L}{\partial \dot{\phi}} &= I_{\perp}\dot{\phi}\sin^2\theta+I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right)\cos\theta, && (18)\\
\frac{\partial L}{\partial \dot{\psi}} &= I_3\left(\dot{\psi}+\dot{\phi}\cos\theta\right). && (19)
\end{aligned}
$$

Differentiating (17), (18) and (19) with respect to time gives:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} &= I_{\perp}\ddot{\theta}, && (20)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} &= \left(I_{\perp}\sin^2\theta+I_3\cos^2\theta\right)\ddot{\phi}+I_3\cos\theta\ddot{\psi} &&\\
&\quad +2\left(I_{\perp}-I_3\right)\sin\theta\cos\theta\dot{\theta}\dot{\phi}, && (21)\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\psi}} &= I_3\left(\ddot{\psi}+\ddot{\phi}\cos\theta-\dot{\phi}\dot{\theta}\sin\theta\right). && (22)
\end{aligned}
$$

Evaluating (11), (12) and (13), some algebraic manipulations yield (2), (3) and (4) as required. □

4.4. Discussion

Examining the equations of motion for the 3D rigid-body model (2) to (4), it can be seen that there is a spin motion, precession and nutation. These equations are non-linear and coupled so characterising the motion is difficult. It is believed that computer simulations would be the best way to examine this system further. Unfortunately, there was not sufficient time to complete computer simulations during the MISG meeting. Once an understanding of the equations of motion have been gained, additional complexity, such as an out-of-balance load, could be added to the model. This method would allow the effects of changes to the model to be assessed.

5. Fluid (wave) analysis

5.1. Introduction

The experimental work showed that the fluid was not stationary relative to the washing machine bowl when excessive vibrations were occurring. This suggested that there was some interaction between the washing machine system (excluding the balance rings) and the fluid in the balance rings such that waves were occurring in the fluid. Equations for the water wave frequency were developed for a simplified system.

5.2. Method

We follow [4]. We wish to study the resonances of the fluid motion under the assumption that the drum is rotating with constant angular velocity about a fixed axis. If such resonance modes exist, it is possible that they could be excited by perturbations to the rotational motion of the drum. If this were to happen, the resulting oscillations could grow in amplitude to those observed for rotational frequencies between 180rpm and 250rpm. Thus our goal is to see if there are any such resonances for rotational frequencies in this range.

5.3. Analysis

Let $\mathbf{q}$ be the velocity of water in the balance rings measured in a co-ordinate system which is fixed relative to the motion of the drum. We assume that the drum is rotating with constant angular velocity $\boldsymbol{\Omega}$. Thus the vector $\boldsymbol{\Omega}$ points in the direction of the axis of rotation (which is vertical) and its length $\omega = |\boldsymbol{\Omega}|$ is the angular speed.

The equations of motion of the fluid in this co-ordinate system are for conservation of mass (water is assumed to be incompressible)

$$\nabla \cdot \mathbf{q} = 0 \tag{23}$$

and conservation of momentum

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q} \cdot \nabla \mathbf{q} + 2\boldsymbol{\Omega} \times \mathbf{q} = -\frac{1}{\rho} \nabla p - \nu \nabla \times (\nabla \times \mathbf{q}). \tag{24}$$

Here ν and ρ are respectively the kinematic viscosity and density of water. The scalar p is the *reduced pressure* which is related to the actual pressure P by the equation

$$p = P + \rho g z - \frac{1}{2} \rho \omega^2 r^2, \tag{25}$$

in which r and z are the radial and axial co-ordinates in a cylindrical co-ordinate system with axis $\boldsymbol{\Omega}$.

5.3.1 Steady State Motion. The ideal state of the system is for the fluid to be stationary relative to the rotating reference frame. In this case, $q = 0$ and (24) shows that $\nabla p = 0$ so that $p =$ constant. Equation (25) gives the actual pressure within the rotating liquid.

On the air-liquid interface within the balance rings, the air pressure P_0 must be the same as the liquid pressure. Thus (25) gives us an equation for the shape of this interface:

$$z = \frac{\omega^2}{2g} r^2 + \text{constant}, \tag{26}$$

showing that the shape is a paraboloid.

When $\omega = 0$ the paraboloid of course degenerates to a plane horizontal surface. But we are more interested in what happens near 200rpm, for which $\omega \approx 21$. The balance rings have a radius of approximately 220mm, so the slope of the interface,

$$\frac{dz}{dr} = \frac{\omega^2 r}{g} \approx 10,$$

is quite steep at such frequencies.

5.3.2 Waves: Equations and boundary conditions. Assuming that the nonlinear and viscous effects of the motion may be ignored, equation (24) becomes

$$\frac{\partial \mathbf{q}}{\partial t} + 2\mathbf{\Omega} \times \mathbf{q} = -\frac{1}{\rho} \nabla p. \tag{27}$$

This is a reasonable approximation provided that the two dimensionless parameters

$$E = \frac{\nu}{\omega L^2} \quad \text{(Ekman number)}, \qquad \epsilon = \frac{U}{\omega L} \quad \text{(Rossby number)},$$

are small. Here, U and L represent typical velocity and length scales for the motion of the liquid.

We seek oscillatory solutions

$$\mathbf{q} = \mathbf{Q} e^{i\lambda t},$$
$$p = \text{constant} + \Phi e^{i\lambda t},$$

where $\mathbf{Q}$ and Φ are complex-valued functions of the spacial coordinates and it is understood that the physical velocity and reduced pressure are the real parts of $\mathbf{q}$ and p.

Substituting these expressions into (23) and (27) gives the equations that must be satisfied by $\mathbf{Q}$ and Φ:

$$\nabla \cdot \mathbf{Q} = 0, \tag{28}$$

$$i\lambda \mathbf{Q} + 2\boldsymbol{\Omega} \times \mathbf{Q} = -\frac{1}{\rho}\nabla\Phi. \tag{29}$$

The normal flux of water at solid boundaries must be zero. Thus if Γ_0 denotes the portion of the inner surface of the balance rings in contact with the water, and if $\mathbf{n}$ denotes the outward unit normal vector on Γ_0 then we must have

$$\mathbf{n}.\mathbf{Q} = 0 \quad \text{on } \Gamma_0. \tag{30}$$

Let Γ_1 denote the water surface in contact with air inside the balance rings. Fluid particles on this surface will stay on this surface so they experience constant pressure $P = P_0$. Thus $dP/dt = 0$ on Γ_1, where d/dt denotes the time derivative following the water,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla.$$

Applying this operator to (25), we find

$$\frac{dp}{dt} = \mathbf{q} \cdot \nabla(\rho g z - \frac{1}{2}\rho\omega^2 r^2) \quad \text{on } \Gamma_1.$$

But the unit normal $\mathbf{n}$ to the steady state surface (26) is given by

$$\mathbf{n} = \frac{\nabla(\rho g z - \frac{1}{2}\rho\omega^2 r^2)}{|\nabla(\rho g z - \frac{1}{2}\rho\omega^2 r^2)|}.$$

Further,

$$\frac{dp}{dt} = \frac{\partial p}{\partial t} + \mathbf{q} \cdot \nabla p \approx \frac{\partial p}{\partial t}.$$

Thus the free surface boundary condition for our linear waves is given by

$$i\lambda\Phi = \Theta \mathbf{n} \cdot \mathbf{Q} \quad \text{on } \Gamma_1, \tag{31}$$

where Γ_1 is given approximately by (26) and

$$\Theta = |\nabla(\rho g z - \frac{1}{2}\rho\omega^2 r^2)| = \rho\sqrt{g^2 + \omega^4 r^2}.$$

We see then that the water waves must satisfy equations (28), (29) and the two boundary conditions (30), (31).

5.3.3 Waves: Rough calculation. Figure 4 shows schematically how the two concentric balance rings are divided into cells. Narrow slots connecting adjacent cells in each ring allow the water to flow around each ring. The slots provide a simple damping mechanism for large scale movement of the fluid in the rings.

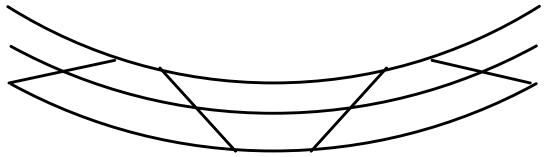

Figure 4. The shape of the cells within balance rings

The videos provided by Fisher & Paykel and the experiments performed at MISG showed strong fluid oscillations within the cells during the resonant motion of the washing machine. The fluid flow for these oscillations appeared to be two dimensional, the direction of motion being perpendicular to the axis of rotation of the drum. The fluid flow from one cell to another seemed to be on a slower time scale (period ≈ 4 seconds) than the oscillations within each cell.

These observations partly justify some simplifying assumptions:

- The flow is two dimensional.
- The oscillations of fluid within each cell are not significantly affected by the flow of fluid through the slots connecting adjacent cells. Hence we ignore the slots in this analysis.

The shape of the cells themselves will affect the oscillation frequency. However, in order to get a rough idea of the nature of these oscillations, we assume a simplified geometry for the cells. This is shown in Figure 5, which also illustrates a local co-ordinate system for a cell.

With respect to this co-ordinate system, equations (28), (29) for $\mathbf{Q} = (Q_1, Q_2, 0)$ and Φ take the form

$$\frac{\partial Q_1}{\partial x} + \frac{\partial Q_2}{\partial y} = 0, \tag{32}$$

$$i\lambda Q_1 + 2\omega Q_2 = -\frac{1}{\rho}\frac{\partial \Phi}{\partial x}, \tag{33}$$

$$i\lambda Q_2 - 2\omega Q_1 = -\frac{1}{\rho}\frac{\partial \Phi}{\partial y}. \tag{34}$$

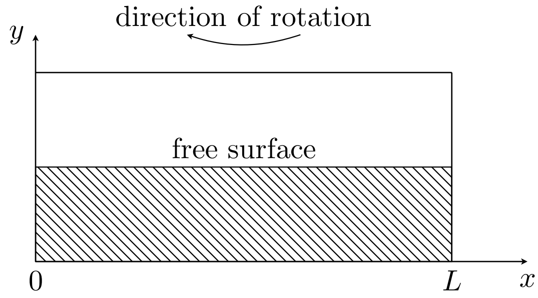

Figure 5. Simplified cell geometry and local co-ordinate system

We noted in Section 5.3.1 that the free surface is almost cylindrical for the large values of ω that interest us. Given that the cells are small, we take this approximation one step further and approximate the steady free surface within a cell by a plane parallel to the axis of rotation. The approximate version of Equation (31) takes the form

$$i\lambda\Phi = \rho\omega^2 R Q_2, \tag{35}$$

where R is the balance ring radius. On the solid boundaries $x = 0$, $x = L$ we must have $Q_1 = 0$ and on $y = 0$ we must have $Q_2 = 0$.

In terms of the dimensionless variables

$$\xi = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad u_1 = \frac{Q_1}{U}, \quad u_2 = \frac{Q_2}{U}, \quad \phi = \frac{\Phi}{\rho\omega U L}, \quad \sigma = \frac{\lambda}{\omega},$$

our equations take the form

$$\frac{\partial u_1}{\partial \xi} + \frac{\partial u_2}{\partial \eta} = 0, \tag{36}$$

$$i\sigma u_1 + 2u_2 = -\frac{\partial \phi}{\partial \xi}, \tag{37}$$

$$i\sigma u_2 - 2u_1 = -\frac{\partial \phi}{\partial \eta}. \tag{38}$$

The boundary conditions are

$$u_1(0,\eta) = u_1(1,\eta) = 0, \quad 0 < \eta < h, \tag{39}$$

$$u_2(\xi,0) = 0, \quad 0 < \xi < 1, \tag{40}$$

$$u_2(\xi,h) = i\frac{\sigma L}{R}\phi(\xi,h), \quad 0 < \xi < 1, \tag{41}$$

where $\eta = h \approx$ constant is the dimensionless equation for the free surface. We may think of h as being the radial depth of the water in a cell divided by the cell's length L.

The flow is two dimensional so the equation of continuity (36) implies the existence of a stream function ψ such that

$$u_1 = -\frac{\partial \psi}{\partial \eta}, \quad u_2 = \frac{\partial \psi}{\partial \xi}.$$

This allows us to write (37), (38) as

$$\frac{\partial}{\partial \xi}(\phi + 2\psi) = i\sigma \frac{\partial \psi}{\partial \eta},$$
$$\frac{\partial}{\partial \eta}(\phi + 2\psi) = -i\sigma \frac{\partial \psi}{\partial \xi}.$$

These equations imply that $i(\phi + 2\psi)/\sigma$ is a complex velocity potential, i.e.

$$u_1 = \frac{i}{\sigma}\frac{\partial}{\partial \xi}(\phi + 2\psi), \quad u_2 = \frac{i}{\sigma}\frac{\partial}{\partial \eta}(\phi + 2\psi).$$

The equations also show that ϕ and ψ satisfy Laplace's equation

$$\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0, \quad \frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \eta^2} = 0.$$

Remark 1 *One could make use of this potential flow. It allows the use of the conformal mapping technique to treat cell geometries that are more realistic than that shown in Figure 5. This technique is discussed in many fluid mechanics and complex analysis texts, such as [3].*

The solid boundary conditions (39) and (40) are satisfied if we take ψ to be zero on the solid boundaries. The free boundary condition (41) takes the form

$$\frac{\partial \psi}{\partial \xi} = \frac{i\sigma L}{R}\phi \quad \text{if } \eta = h. \tag{42}$$

The technique of separation of variables furnishes an infinite sequence of solutions that satisfy the solid boundary conditions:

$$\psi_n(\xi, \eta) = \sin(n\pi\xi)\sinh(n\pi\eta)$$
$$\phi_n(\xi, \eta) = -i\sigma\cos(n\pi\xi)\cosh(n\pi\eta) - 2\sin(n\pi\xi)\sinh(n\pi\eta),$$

for $n = 1, 2, \ldots, \infty$. We must take an infinite linear combination of these functions to find solutions that satisfy the free boundary condition

$$\psi(\xi, \eta) = \sum_{n=1}^{\infty} \frac{b_n}{\sinh(n\pi h)}\psi_n(\xi, \eta), \quad \phi(\xi, \eta) = \sum_{n=1}^{\infty} \frac{b_n}{\sinh(n\pi h)}\phi_n(\xi, \eta),$$

where the $\sinh(n\pi h)$ terms in the denominators are introduced to simplify the following calculations.

The free boundary condition (42) implies that

$$\sum_{n=1}^{\infty} n\pi b_n \cos(n\pi\xi) = \sum_{n=1}^{\infty} \frac{\sigma^2 L}{R} b_n \coth(n\pi h)\cos(n\pi\xi) - \frac{2i\sigma L}{R} b_n \sin(n\pi\xi), \quad 0 < \xi < 1. \tag{43}$$

We are interested in non-trivial solutions of this equation and such solutions will only exist for certain values of the eigenvalue parameter σ. It is more convenient to work with an equivalent equation obtained by multiplying (43) by $\cos(m\pi\xi)$ and integrating the resulting equation over $[0, 1]$:

$$\left(m\pi - \frac{\sigma^2 L}{R}\coth(m\pi h)\right) b_m = \frac{-8i\sigma L}{R}\sum_{n=1}^{\infty} b_n \Gamma_{mn}, \quad m = 1, 2, \ldots, \infty, \tag{44}$$

where

$$\Gamma_{mn} = \begin{cases} 0, & \text{if } n+m \text{ is even,} \\ \frac{n}{\pi(n^2-m^2)}, & \text{if } n+m \text{ is odd.} \end{cases}$$

For computational purposes it is sufficient to take $1 \le m, n \le N$ for some sufficiently large value of N. If this is done then the non-dimensional eigenfrequencies σ are simply the zeros of the determinant of the coefficient matrix for the equations giving the constants b_m. We used this procedure to plot the first three eigenfrequencies against relative radial depth h in Figure 6.

Figure 6 indicates that for certain values of h, we can expect relative eigenfrequencies close to 1. This corresponds to the liquid oscillating with the rotational frequency ω, which is the predominant forcing frequency of the system. Thus we cannot rule out the possibility that there is a resonance between the cellular oscillations of the liquid and the forced motion of the drum which leads to the unwanted motion. We note that we have not shown that this is necessarily the cause of the unwanted motion.

6. Conclusions

- Modern washing machines with balance rings are complicated systems and it was not possible to build a 3D model including balance rings and an out-of-balance load in the time available.

- [2] showed that balance rings can reduce eccentricity due to an out-of-balance-load but cannot eliminate it.

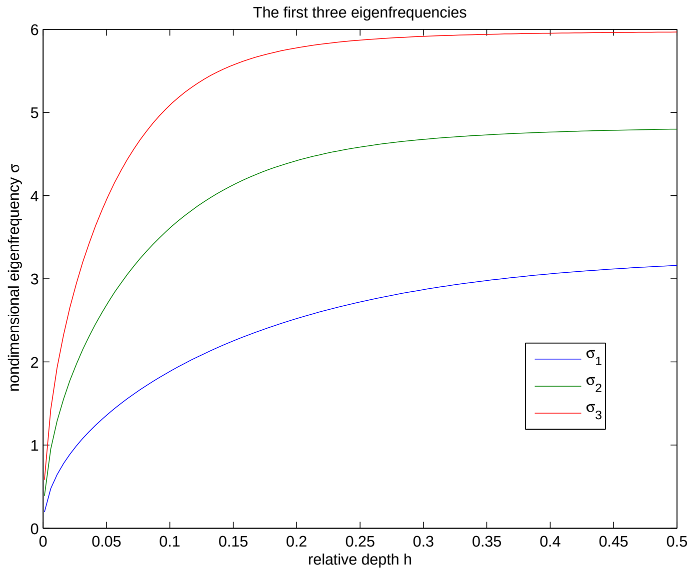

Figure 6. The first 3 eigenfrequencies as a function of relative radial depth h. We used $L/R = 2\pi/24$ which corresponds to an average relative cell length when there are 24 cells.

- A 3D model of a washing machine without balance rings, and without an out-of-balance-load showed that the motion can be described as a combination of spin, precession and nutation.

- The equations of motion for the simplified 3D model were non-linear. Furthermore, linearising the equations removed the motion of interest (precession and nutation).

- It is clear that the motion of the water in the balance rings exacerbates the resonant behaviour of the washing machine. It is also clear that this is because the water moves away from its balancing state, possibly even adding to the effect of the out of balance load. Our analysis of water oscillations within the balance ring does not rule out the possibility that the oscillations themselves start the unwanted motion. However, further work is needed to see if this is the case.

- A good starting point for future research on the effect of the balance rings is Equation 33, which for low frequency ($\lambda \approx 0$) oscillations becomes

$$2\omega Q_2 = -\frac{1}{\rho}\frac{\partial \Phi}{\partial x}.$$

This shows that motion of the balance ring in the y-direction induces a pressure gradient in the x-direction. It is conceivable that such pressure gradients cause pressure jumps between adjacent cells which force the motion of water from one cell to another. It would be interesting to analyse this effect more closely to see if it is the mechanism for the onset of the resonance.

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